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ALTERNATIVE PROCEDURE FOR VISCOELASTIC ANALYSIS OF PLATES BY THE BOUNDARY ELEMENT METHOD

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Abstract

This study presents an alternative Boundary Element formulation for the analysis of viscoelastic plate in bending without using convolution processes with internal cells, Laplace transforms or special fundamental solutions. Two different constitutive models are considered. The first and simplest one is the Kelvin-Voigt model that does not take into account instantaneous response. The second, Boltzmann model, considers instantaneous and time dependent behavior of materials. An appropriate kinematical relation is combined with differential viscoelastic constitutive representations in order to generate the time marching scheme. Spatial approximations are used for boundary elements before any time solution. The proposed technique results in a time marching process that does not use relaxation functions to recover viscous behavior. Some examples are shown in order to demonstrate the accuracy and stability of the technique.

Keywords: Viscoelasticity. Boundary Elements. Numerical time integration.

1 INTRODUCTION

Various engineering structures are constituted by plates in bending and the good representation of these elements for general material or subjected to any load condition is necessary. One of these situations is related to the material viscoelastic behavior as for example, polymers, concrete, wood and others.

Usual numerical viscoelastic analysis are based on relaxation functions [1-4] together with a convenient incremental scheme where the convolutional aspect of the viscous behavior is transformed into discrete contributions to the elastic response. These incremental techniques calculate viscous residuals by local (point by point) stress decay considerations, like viscoplastic processes [5-7], usually requiring cells or other related process to do the domain integrals.

Other possibilities are also present in literature, as doing Laplace transforms or using fundamental solutions for viscoelasticity. The last strategy depends upon the existence of fundamental solutions for each class of problem to be solved.

An alternative procedure to solved viscoelastic problems has been proposed and successfully tested for two and three dimensional solids by [8-12]. This strategy is based on static fundamental solutions and differential viscoelastic constitutive relations that provide accurate results without using domain integrals and with small computational effort.

Encouraged by the absence of cells and the small amount of computations the authors extended here the previous procedure to treat viscoelastic plate in bending problems by the boundary element method. The adopted kinematics is the Kirchhoff one and the constitutive relations are the Kelvin-Voigt and the Boltzmann ones.

An important characteristic of the proposed technique is that the experimental results for creep and relaxation functions can be used to achieve the necessary viscous parameters used in the differential constitutive relations.

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At the end of the paper, selected examples are shown in order to demonstrate the accuracy and stability of the formulation. Along all text Einstein notation is adopted.

2 RHEOLOGICAL MODELS

2.1 Differential representation of the Kelvin-Voigt

The Kelvin-Voigt viscoelastic model can be represented by the simple parallel arrangement of a spring and a dashpot, as depicted in figure 1.

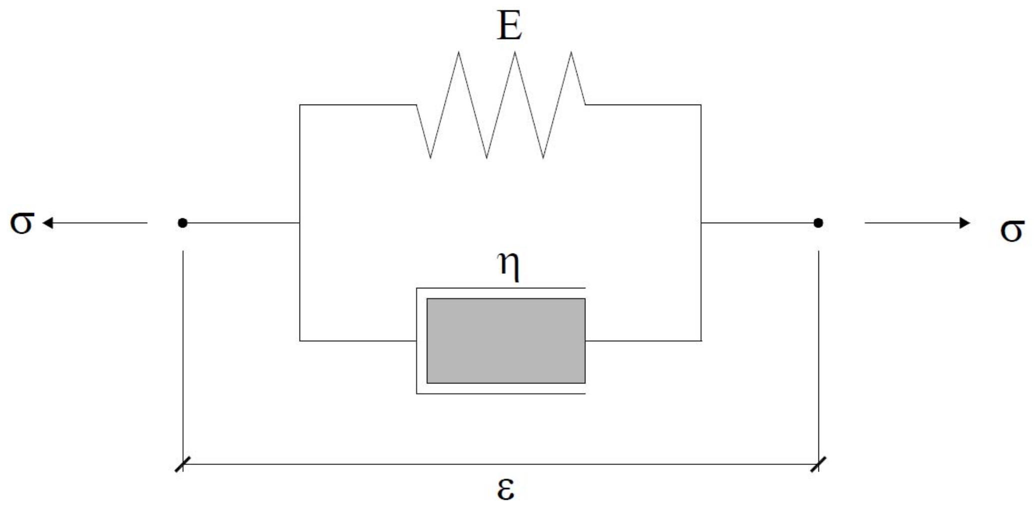

Figure 1 – Kelvin-Voigt model.

The two parts of this model develop the same strain, i.e:

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} = \varepsilon_{ij}^{v} \tag{1}$$

where ε_{ij} is the total strain, ε_{ij}^{e} is the elastic strain ε_{ij}^{v} is the viscous strain tensor..

However the total stress developed in the arrangement is the summation of the viscous and elastic parts, as

$$\sigma_{ij} = \sigma_{ij}^{e} + \sigma_{ij}^{v} \tag{2}$$

where σ_{ij} is the total stress, σ_{ij}^{e} is the elastic stress and σ_{ij}^{v} is the viscous stress.

The elastic and viscous stress are related to strain as follows:

$$\sigma_{ij}^{e} = C_{ij}^{lm}\varepsilon_{lm}^{e} = C_{ij}^{lm}\varepsilon_{lm} \tag{3}$$

$$\sigma_{ij}^{v} = \eta_{ij}^{lm}\dot{\varepsilon}_{lm}^{v} = \eta_{ij}^{lm}\dot{\varepsilon}_{lm} \tag{4}$$

where C_{ij}^{lm} is the elastic strain, $\dot{\varepsilon}_{lm}$ is the time strain rate and η_{ij}^{lm} is the viscous constitutive tensor.

The fourth order tensors C_{ij}^{lm} and η_{ij}^{lm} are given by:

$$C_{ij}^{lm} = \lambda\delta_{ij}\delta_{lm} + \mu\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right) \tag{5}$$

$$\eta_{ij}^{lm} = \theta_{\lambda}\lambda\delta_{ij}\delta_{lm} + \theta_{\mu}\mu\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right) \tag{6}$$

where λ e μ are the elastic constants given by:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{7}$$

$$\mu = G = \frac{E}{2(1+\nu)} \tag{8}$$

θ_λ and θ_μ are material viscous coefficients.

The viscosity tensor η_{ij}^{lm} can be simplified adopting a unique viscous parameter γ as:

$$\gamma = (\theta_{\lambda} + \theta_{\mu})/2 \tag{9}$$

$$\eta_{ij}^{lm} = \gamma\left[\lambda\delta_{ij}\delta_{lm} + \mu\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\right] = \gamma C_{ij}^{lm} \tag{10}$$

Introducing equations (3) and (10) into equation (2), one finds:

$$\sigma_{ij} = C_{ij}^{lm}\varepsilon_{lm} + \gamma C_{ij}^{lm}\dot{\varepsilon}_{lm} \tag{11}$$

2.2 Differential representation of the Boltzmann model

The Boltzmann model is represented by a serial arrangement between an elastic part and the Kelvin-Voigt model, as described by figure 2.

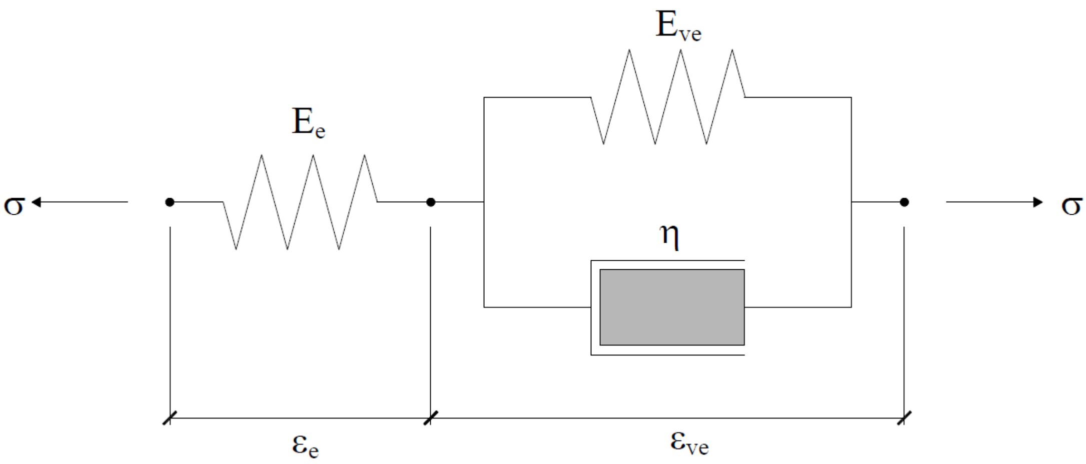

Figure 2 – Boltzmann Model.

The elastic part of the model is responsible by the instantaneous response of the material. The stress at all parts of the model is the same, therefore:

$$\sigma_{ij} = \sigma^{e}_{ij} = \sigma^{ve}_{ij} \tag{12}$$

where σ_{ij} is the total stress tensor, σ^{e}_{ij} is the elastic stress and σ^{ve}_{ij} is the viscoelastic one.

For this model the total strain ε_{ij} is the summation of elastic (instantaneous) strain ε^{e}_{ij} and the viscoelastic strain ε^{ve}_{ij}, i.e.:

$$\varepsilon_{lm} = \varepsilon^{e}_{lm} + \varepsilon^{ve}_{lm} \tag{13}$$

By simplicity, one considers the same poison ratio for both parts of the model and, as for the Kelvin-Voigt model, only one viscous parameter. From these assumptions one writes:

$$\sigma^{e}_{ij} = \tilde{C}^{lm}_{ij}\varepsilon^{e}_{lm} = E_{e}C^{lm}_{ij}\varepsilon^{e}_{lm} \tag{14}$$

$$\sigma^{el}_{ij} = \hat{C}^{lm}_{ij}\varepsilon^{ve}_{lm} = E_{ve}C^{lm}_{ij}\varepsilon^{ve}_{lm} \tag{15}$$

$$\sigma^{v}_{ij} = \eta^{lm}_{ij}\dot{\varepsilon}^{ve}_{lm} = \gamma E_{ve}C^{lm}_{ij}\dot{\varepsilon}^{ve}_{lm} \tag{16}$$

where σ^{el}_{ij} is the stress acting at the spring parallel to the dashpot at the viscous part of the Boltzmann model, σ^{v}_{ij} is the viscous stress at the dashpot, E_{e} is the elastic modulus at the instantaneous part of the arrangement, E_{ve} is the elastic modulus at the viscous part of the model, $\tilde{C}^{lm}_{ij}$ and $\hat{C}^{lm}_{ij}$ are elastic tensors written regarding E_{e} and E_{ve}.

The auxiliary tensor C_{ij}^{lm} is written without unity, resulting:

$$C_{ij}^{lm} = \bar{\lambda}\delta_{ij}\delta_{lm} + \bar{\mu}\left(\delta_{ij}\delta_{jm} + \delta_{im}\delta_{jl}\right) \tag{17}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are nondimensional versions of Lamé constants, i.e.:

$$\bar{\lambda} = \frac{\nu}{(1+\nu)(1-2\nu)} \tag{18}$$

$$\bar{\mu} = G = \frac{1}{2(1+\nu)} \tag{19}$$

For the viscoelastic part one writes:

$$\sigma_{ij} = \sigma_{ij}^{ve} = \sigma_{ij}^{el} + \sigma_{ij}^{v} = E_{ve}C_{ij}^{lm}\varepsilon_{lm}^{ve} + \gamma E_{ve}C_{ij}^{lm}\dot{\varepsilon}_{lm}^{ve} \tag{20}$$

Differentiating equation (13) regarding time results the following relation:

$$\dot{\varepsilon}_{lm} = \dot{\varepsilon}_{lm}^{e} + \dot{\varepsilon}_{lm}^{ve} \tag{21}$$

where $\dot{\varepsilon}_{lm}$ is the total time strain rate, $\dot{\varepsilon}_{lm}^{e}$ is the elastic time strain rate and $\dot{\varepsilon}_{lm}^{ve}$ is the viscoelastic time strain rate.

One isolates the elastic and viscoelastic strains from equations (14) and (20) resulting:

$$\varepsilon_{lm}^{e} = \frac{1}{E_{e}}C_{lm}^{ij\,-1}\sigma_{ij} \tag{22}$$

$$\varepsilon_{lm}^{ve} = \frac{1}{E_{ve}}C_{lm}^{ij\,-1}\sigma_{ij} - \gamma\dot{\varepsilon}_{lm}^{ve} \tag{23}$$

Applying equation (21) into (23), one finds:

$$\varepsilon_{lm}^{ve} = \frac{1}{E_{ve}}C_{lm}^{ij\,-1}\sigma_{ij} - \gamma\left(\dot{\varepsilon}_{lm} - \dot{\varepsilon}_{lm}^{e}\right) \tag{24}$$

Using equations (22) and (24) into (13), the rheological differential representation for the Boltzmann model results:

$$\sigma_{ij} = \frac{E_{e}E_{ve}}{E_{e}+E_{ve}}C_{ij}^{lm}\left(\varepsilon_{lm} + \gamma\dot{\varepsilon}_{lm}\right) - \frac{\gamma E_{ve}}{E_{e}+E_{ve}}\dot{\sigma}_{ij} \tag{25}$$

where $\dot{\sigma}_{ij}$ is the total time stress rate.

3 BOUNDARY ELEMENT FORMULATION FOR VISCOELASTC PLATES IN BENDING

3.1 Kelvin-Voigt formulation

The Boundary Element formulation is achieved here from the Betti's reciprocal theorem,

$$\int_V \sigma_{ij}^{*}\varepsilon_{ij} dV = \int_V \sigma_{ij}\varepsilon_{ij}^{*} dV \tag{26}$$

where σ_{ij}^{*} and ε_{ij}^{*} are the static fundamental values. Applying the elastic constitutive equation for σ_{ij}^{*} and the viscoelastic constitutive equation (11) for σ_{ij} one writes:

$$\int_V C_{ij}^{lm}\varepsilon_{lm}^{*}\varepsilon_{ij} dV = \int_V C_{ij}^{lm}\varepsilon_{lm}\varepsilon_{ij}^{*} + \gamma C_{ij}^{lm}\dot{\varepsilon}_{lm}\varepsilon_{ij}^{*} dV \tag{27}$$

Using the Kirchhoff strain displacement relation for plate in bending,

$$\varepsilon_{ij} = -x_3 w,_{ij} \tag{28}$$

$$\dot{\varepsilon}_{ij} = -x_3 \dot{w},_{ij} \tag{29}$$

in equation (27) results

$$\int_V w_{,lm}^{*}(C_{ij}^{lm}x_3^2 w_{,ij})dV = \int_V w_{,ij}(C_{ij}^{lm}x_3^2 w_{,lm}^{*})dV + \int_V \dot{w}_{,ij}(C_{ij}^{lm}x_3^2 w_{,lm}^{*})dV \tag{30}$$

Integrating equation (30) along the thickness of the plate one achieves:

$$\int_\Omega M_{ij} w,_{ij}^{*} \, d\Omega = \int_\Omega M_{ij}^{*} w,_{ij} \, d\Omega + \int_\Omega \gamma M_{ij}^{*}\dot{w},_{ij} \, d\Omega \tag{31}$$

Applying the divergence theorem over equation (31) and taking into account the special concentrated forces at corners of the Kirchhoff theory of plates, the following boundary integral equation for internal collocations is achieved for the Kelvin-Voigt model:

$$\begin{aligned} & w(q) + \gamma\dot{w}(q) + \int_\Gamma \left(V_n^{*}(q,P)w(P) - M_n^{*}(q,P)\frac{\partial w}{\partial n}(P) \right) d\Gamma(P) + \\ & \gamma\int_\Gamma \left(V_n^{*}(q,P)\dot{w}(P) - M_n^{*}(q,P)\frac{\partial \dot{w}}{\partial n}(P) \right) d\Gamma(P) + \sum_{i=1}^{N_c} R_{ci}^{*}(q,P)w_{ci}(P) + \\ & +\gamma\sum_{i=1}^{N_c} R_{ci}^{*}(q,P)\dot{w}_{ci}(P) = \int_\Gamma \left(V_n(P)w^{*}(q,P) - M_n(P)\frac{\partial w^{*}}{\partial n}(q,P) \right) d\Gamma(P) + \\ & +\sum_{i=1}^{N_c} R_{ci}(P)w_{ci}^{*}(q,P) + \int_{\Omega_g} g(p)w^{*}(q,p)d\Omega_g(p) \end{aligned} \tag{32}$$

where q is an internal collocation and P is a boundary Field point.

Taking into account that the singularities of the kernels related to viscous quantities are exactly the same as the static ones the boundary integral equation for collocations Q placed over the boundary is:

$$\begin{aligned}
&K(Q)w(Q)+\gamma K(Q)\dot{w}(Q)+\int_{\Gamma}\left(V_n^*(Q,P)w(P)-M_n^*(Q,P)\frac{\partial w}{\partial n}(P)\right)d\Gamma(P)+\\
&\gamma\int_{\Gamma}\left(V_n^*(Q,P)\dot{w}(P)-M_n^*(Q,P)\frac{\partial \dot{w}}{\partial n}(P)\right)d\Gamma(P)+\sum_{i=1}^{N_c}R_{ci}^*(Q,P)w_{ci}(P)+\\
&+\gamma\sum_{i=1}^{N_c}R_{ci}^*(Q,P)\dot{w}_{ci}(P)=\int_{\Gamma}\left(V_n(P)w^*(Q,P)-M_n(P)\frac{\partial w^*}{\partial n}(Q,P)\right)d\Gamma(P)+\\
&+\sum_{i=1}^{N_c}R_{ci}(P)w_{ci}^*(Q,P)+\int_{\Omega_g}g(p)w^*(Q,p)d\Omega_g(p)
\end{aligned}\tag{33}$$

where K(Q) is the usual free term for elastic plates. The curvature for internal points is given by differentiating twice equation (32) regarding the collocation position. As the dependence of all kernels regarding the collocation point position is exactly the same as the usual static formulation, the result is achieved directly, i.e.:

$$\begin{aligned}
&\frac{\partial^2 w(q)}{\partial x_k\partial x_l}+\gamma\frac{\partial^2 \dot{w}(q)}{\partial x_k\partial x_l}+\int_{\Gamma}\left(\frac{\partial^2 V_n^*}{\partial x_k\partial x_l}(q,P)w(P)-\frac{\partial^2 M_n^*}{\partial x_k\partial x_l}(q,P)\frac{\partial w}{\partial n}(P)\right)d\Gamma(P)+\\
&+\gamma\int_{\Gamma}\left(\frac{\partial^2 V_n^*}{\partial x_k\partial x_l}(q,P)\dot{w}(P)-\frac{\partial^2 M_n^*}{\partial x_k\partial x_l}(q,P)\frac{\partial \dot{w}}{\partial n}(P)\right)d\Gamma(P)+\\
&+\sum_{c=1}^{N_c}\frac{\partial^2 R_c^*}{\partial x_k\partial x_l}(q,P)w_c(P)+\gamma\sum_{c=1}^{N_c}\frac{\partial^2 R_c^*}{\partial x_k\partial x_l}(q,P)\dot{w}_c(P)=\\
&=\int_{\Gamma}\left(V_n(P)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,P)-M_n(P)\frac{\partial^2}{\partial x_k\partial x_l}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)d\Gamma(P)+\\
&+\sum_{c=1}^{N_c}R_c(P)\frac{\partial^2 w_c^*}{\partial x_k\partial x_l}(q,P)+\int_{\Omega_g}g(p)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,p)d\Omega_g(p)
\end{aligned}\tag{34}$$

The total internal moment is achieved applying the viscoelastic constitutive relation written for plates from equation (11) as:

$$M_{ij}=-\int_{-h/2}^{h/2}(C_{ij}^{lm}w_{,lm}+\gamma C_{ij}^{lm}\dot{w}_{,lm})x_3^2dx_3=M_{ij}^e+M_{ij}^v \tag{35}$$

and by the application of the elastic tensor results,

$$M_{ij}=-D\left[\nu w,_{kk}\delta_{ij}+(1-\nu)w,_{ij}\right]+\gamma D\left[\nu\dot{w},_{kk}\delta_{ij}+(1-\nu)\dot{w},_{ij}\right]=M_{ij}^e+M_{ij}^v \tag{36}$$

In order to calculate the transverse internal force one has to differentiate equation (35) regarding the collocation position, as follows

$$
\begin{aligned}
&\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w(q)}{\partial x_m \partial x_m}\right)+\gamma\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 \dot{w}(q)}{\partial x_m \partial x_m}\right)+\int_\Gamma\left(\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 V_n^*}{\partial x_m \partial x_m}(q,P)\right)w(P)+\right.\\
&-\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 M_n^*}{\partial x_m \partial x_m}(q,P)\right)\frac{\partial w}{\partial n}(P)\bigg)d\Gamma(P)+\gamma\int_\Gamma\left(\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 V_n^*}{\partial x_m \partial x_m}(q,P)\right)\dot{w}(P)+\right.\\
&-\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 M_n^*}{\partial x_m \partial x_m}(q,P)\right)\frac{\partial \dot{w}}{\partial n}(P)\bigg)d\Gamma(P)+\sum_{c=1}^{N_c}\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 R_c^*}{\partial x_m \partial x_m}(q,P)\right)w_c(P)+\\
&+\gamma\sum_{c=1}^{N_c}\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 R_c^*}{\partial x_m \partial x_m}(q,P)\right)\dot{w}_c(P)=\int_\Gamma\left(V_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_m \partial x_m}(q,P)\right)+\right.\\
&-M_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2}{\partial x_m \partial x_m}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)\bigg)d\Gamma(P)+\\
&+\sum_{c=1}^{N_c}R_c(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w_c^*}{\partial x_m \partial x_m}(q,P)\right)+\int_{\Omega_g}g(p)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_m \partial x_m}(q,p)\right)d\Omega_g(p)
\end{aligned}
\tag{37}
$$

and use the following relation

$$q_j=-D(w,_{kkj}+\gamma\dot{w},_{kkj}) \tag{38}$$

Using boundary elements approximation for Kirchhoff plate in bending formulation, the integral equation (33) is expressed in a matrix form as:

$$
\begin{aligned}
&K(Q)u(Q)+\gamma K(Q)\dot{u}(Q)+\bar{H}(Q)\bar{U}+\gamma\bar{H}(Q)\dot{\bar{U}}+\bar{H}_c(Q)\bar{w}_c+\\
&+\gamma\bar{H}_c(Q)\dot{\bar{w}}_c=\bar{G}(Q)\bar{P}+\bar{G}_c(Q)\bar{R}_c+\bar{T}(Q)
\end{aligned}
\tag{39}
$$

where $\bar{H}(Q)$ and $\bar{G}(Q)$ are matrices that contain the contribution of internal fundamental efforts and fundamental displacements, respectively. $\bar{H}_c(Q)$ and $\bar{G}_c$ contain the concentrated reactions and corner displacements, respectively. $T(Q)$ is the distributed force vector, $\bar{U}$ is the displacement vector at the boundary, $\dot{\bar{U}}$ is the velocity vector at the boundary, $\bar{P}$ is the force vector at the boundary. The corner displacement, velocity and force are, respectively, $\bar{w}_c$, $\dot{\bar{w}}_c$ and $\bar{R}_c$.

General displacement and velocity vectors also contain the usual normal rotation as:

$$
\begin{aligned}
&\bar{U}=\left\{w^1 \quad \frac{\partial w}{\partial n}^1 \quad w^2 \quad \frac{\partial w}{\partial n}^2 \quad \ldots \quad w^{N_n} \quad \frac{\partial w}{\partial n}^{N_n}\right\}\\
&\dot{\bar{U}}=\left\{\dot{w}^1 \quad \frac{\partial \dot{w}}{\partial n}^1 \quad \dot{w}^2 \quad \frac{\partial \dot{w}}{\partial n}^2 \quad \ldots \quad \dot{w}^{N_n} \quad \frac{\partial \dot{w}}{\partial n}^{N_n}\right\}
\end{aligned}
\tag{41}
$$

In the same way the boundary force vector is given by:

$$\overline{P} = \left\{ V_n^1 \quad M_n^1 \quad V_n^2 \quad M_n^2 \quad \dots \quad V_n^{N_n} \quad M_n^{N_n} \right\} \tag{42}$$

Corner displacement, velocity and force are given as:

$$\begin{aligned} \overline{w}_c &= \left\{ w_c^1 \quad w_c^2 \quad \dots \quad w_c^{N_c} \right\} \\ \dot{\overline{w}}_c &= \left\{ \dot{w}_c^1 \quad \dot{w}_c^2 \quad \dots \quad \dot{w}_c^{N_c} \right\} \\ \overline{R}_c &= \left\{ R_c^1 \quad R_c^2 \quad \dots \quad R_c^{N_c} \right\} \end{aligned} \tag{43}$$

Finally vectors $u(Q)$ and $\dot{u}(Q)$ contain the transverse displacement at boundary colocations, i.e.:

$$\begin{aligned} u &= \left\{ w^1 \quad w^2 \quad \dots \quad w^{N_n} \right\} \\ \dot{u} &= \left\{ \dot{w}^1 \quad \dot{w}^2 \quad \dots \quad \dot{w}^{N_n} \right\} \end{aligned} \tag{44}$$

All vectors dimensions are indicated by superscrits N_n and N_c that are nodal and corner points, respectively.

In this work linear functions are used to approximate the geometry while quadratic functions are used to approximate variables. Special schemes are employed to distribute $K(Q)$ over $\overline{H}(Q)$. Therefore the corner reaction $\overline{R}_c$ is settled null and the corner displacement $\overline{w}_c$ becomes a function of neighbor nodes. These arrangements are also employed to viscous terms.

From the above considerations one achieves the following system of time differential equations for the Kelvin-Voigt viscoelastic problem.

$$\overline{H}\overline{U} + \gamma \overline{H}\dot{\overline{U}} = \overline{G}\overline{P} + \overline{T} \tag{45}$$

The displacement equations for internal points, following the same reasoning is given by:

$$\overline{u}(q) + \gamma \dot{\overline{u}}(q) + \overline{H}'\overline{U} + \gamma \overline{H}'\dot{\overline{U}} = \overline{G}'\overline{P} + \overline{T}' \tag{46}$$

For curvature and internal efforts similar equations can be found.

Time integration should be done in order to solve equation (45). All the time derivatives are of the first order with constant coefficients it is enough to adopt a simple linear approximation over a time step as:

$$\dot{w} = \frac{w_{(s+1)} - w_{(s)}}{\Delta t} \tag{47}$$

$$\frac{\partial \dot{w}}{\partial n} = \frac{\frac{\partial w}{\partial n}_{(s+1)} - \frac{\partial w}{\partial n}_{(s)}}{\Delta t} \qquad (48)$$

Or, following a general notation,

$$\dot{\bar{U}} = \frac{\bar{U}_{(s+1)} - \bar{U}_{(s)}}{\Delta t} \qquad (49)$$

Substituting equation (49) into equation (45) results:

$$\tilde{H}\bar{U}_{(s+1)} = \bar{G}\bar{P}_{(s+1)} + \bar{T} + \tilde{F}_s \qquad (50)$$

where:

$$\tilde{H} = \left(1 + \frac{\gamma}{\Delta t}\right)\bar{H} \qquad (51)$$

$$\tilde{F}_s = \frac{\gamma}{\Delta t}\bar{H}\bar{U}_{(s)} \qquad (52)$$

It is important to note that $\tilde{F}_s$ is a known value of the past and the boundary conditions are imposed by simply changing rows of matrices $\tilde{H}$ and $\bar{G}$ and summing the independent vectors $\tilde{F}$ and $\bar{T}$. The solution of equation (50) gives the current displacement and reactions.

From present and past displacements values one calculates velocity by equation (49).

In order to solve internal values, equations (46) that includes displacements, curvatures and curvature derivatives one applies approximations (47) and

$$\dot{w},_{ij} = \frac{w,_{ij(s+1)} + w,_{ij(s)}}{\Delta t} \qquad (53)$$

$$\dot{w},_{kk\beta} = \frac{w,_{kk\beta(s+1)} + w,_{kk\beta(s)}}{\Delta t} \qquad (54)$$

What can be summarized as,

$$\dot{\bar{u}} = \frac{\bar{u}_{(s+1)} - \bar{u}_{(s)}}{\Delta t} \qquad (55)$$

The next step is important to solve the viscolastic problems by differential procedures, i.e., applying equation (55) on equation (46) results:

$$\bar{u}_{(s+1)} = \left(-\bar{H}'\bar{U} - \gamma\bar{H}'\dot{\bar{U}} + \bar{G}'\bar{P} + \bar{T}' + \frac{\gamma}{\Delta t}u_{(s)} \right) \Big/ \left(1 + \frac{\gamma}{\Delta t} \right) \tag{56}$$

Substituting the boundary displacements and velocities into (56) the values of internal displacements, curvatures and curvature derivatives are calculated. As a consequence all velocities are calculated for intenal points applying equations (47), (53) e (54). It is important to note that expressions (1) to (4) can be rewritten for plate analysis as follows:

$$w,_{ij} = w,_{ij}^{e} = w,_{ij}^{v} \tag{57}$$

$$M_{ij} = M_{ij}^{e} + M_{ij}^{v} \tag{58}$$

$$M_{ij}^{e} = -D\left[\nu w,_{kk}\delta_{ij} + (1-\nu)w,_{ij} \right] \tag{59}$$

$$M_{ij}^{v} = -\gamma D\left[\nu \dot{w},_{kk}\delta_{ij} + (1-\nu)\dot{w},_{ij} \right] \tag{60}$$

$$q_{\beta} = -D\left(w,_{kk\beta} + \dot{w},_{kk\beta} \right) \tag{61}$$

These equations complete the procedure for the Kelvin model as all internal efforts are achieved.

3.1.1 Clamped supported square plate

The analysis consists of a square plate with two simple supported opposite sides and the other two clamped. The material follows the Kelvin-Voigt viscoelastic model. This plate has been discretized by 24 boundary elements. (Figure 3). The Physical parameters used for this analysis are: $E = 2.5 \mathrm{x} 10^{7}\ \mathrm{kN/m^2}$, $\nu = 0.3$, $\gamma = 7.14285\,\mathrm{days}$, $q = 10\mathrm{kN/m^2}$, $a = 3\mathrm{m}$, $t = 0{,}06\mathrm{m}$ and $\Delta t = 0{,}1\mathrm{day}$. Twenty four boundary elements with 1000 time steps are adopted to run this example.

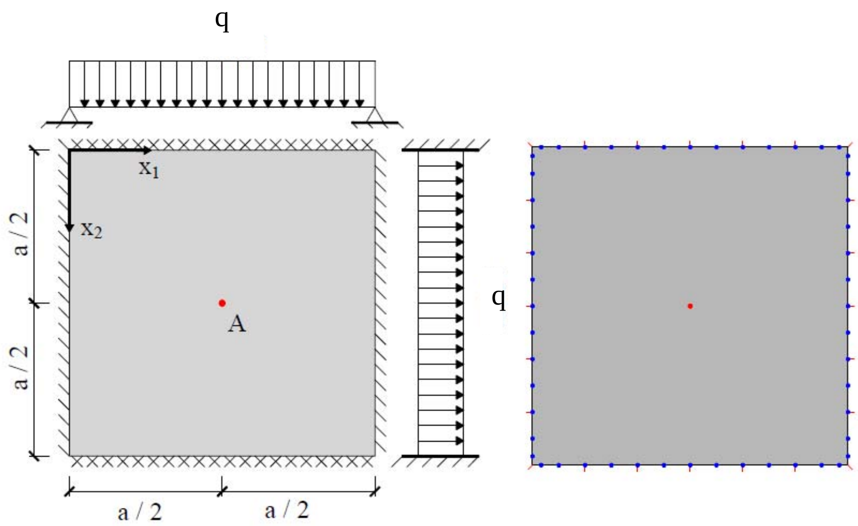

Figure 3 – Geometry, and discretization.

The results for curvature w,11 and w,22 along time for point A are depicted in figure 4. Figure 5 shows results for M22.

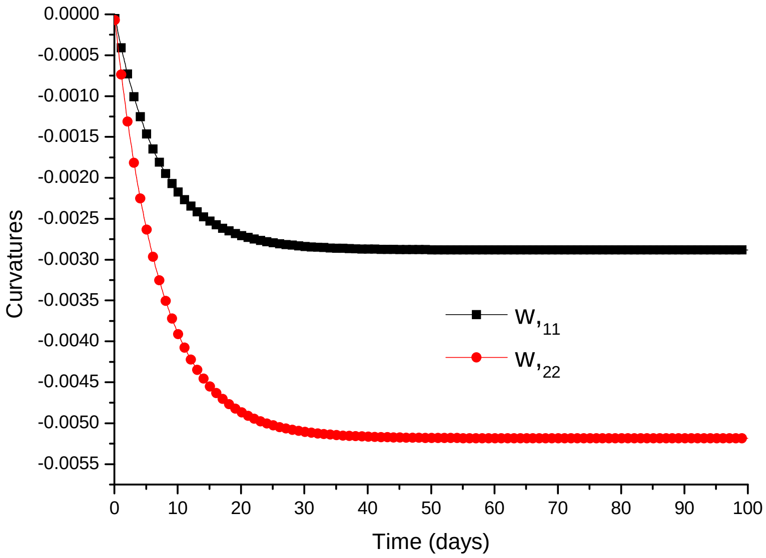

Figure 4 – Curvatures along time for point A.

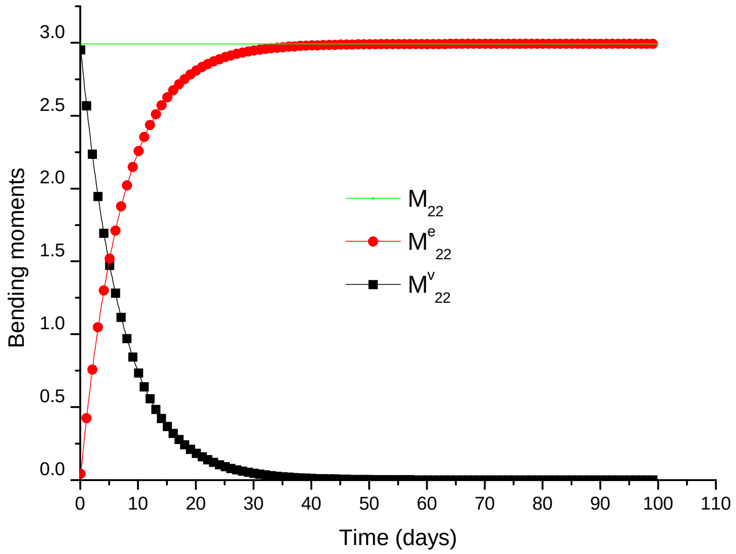

Figure 5 – Noments along time for point A.

One may observe that about 50 days the final values of displacements and internal forces are practically achieved. In figure 5 one may observe the transfer from viscous stresses to elastic stress as the time goes by. Moreover, the sum between the elastic and viscous part results exactly the total static moment. It I important to observe that the total moment achieved by the viscoelastic solution is exactly the same achieved by a pure elastic analysis.

3.1.2 Corner clamped-free plate

Adopting the same geometry and physical parameters of the previous example, one considers a plate clamped at two adjacent sides and free at the other two sides, see figure 6. This example is used to show the solution behavior for different time steps.

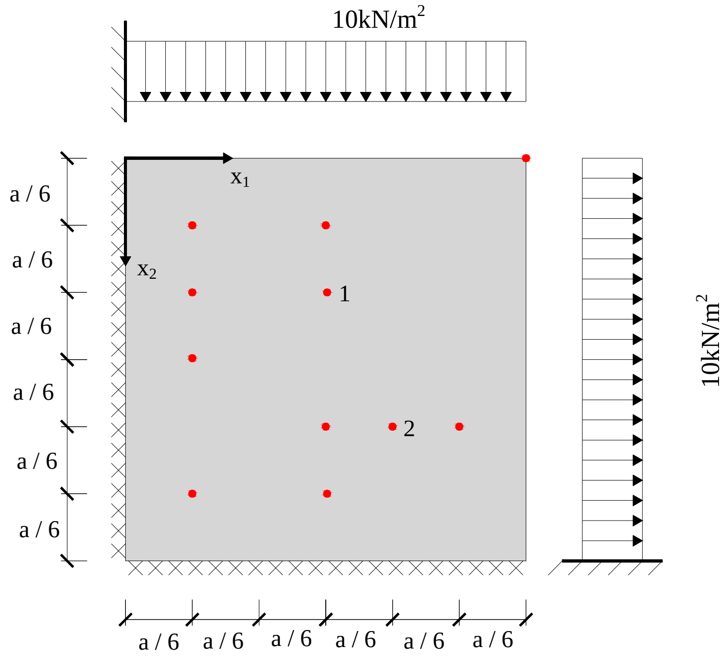

Figure 6 – Geometry and applied load.

Using the same discretization one can see in figures 6 and 7 the influence of time step on the solution for displacement at point 1 and curvature $w_{,xx}$ at point 2.

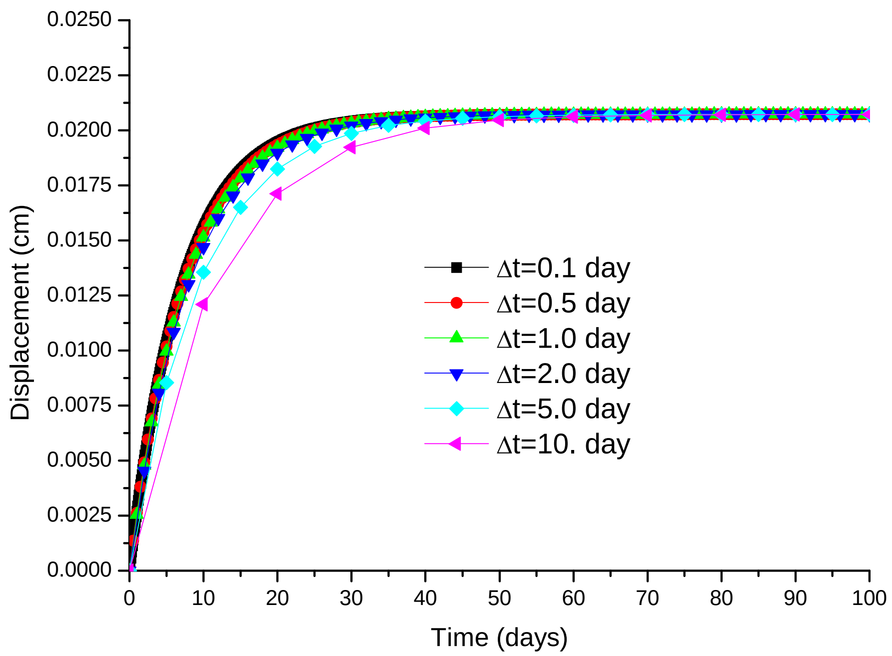

Figure 7 – Displacement at point 1 – time step dependence.

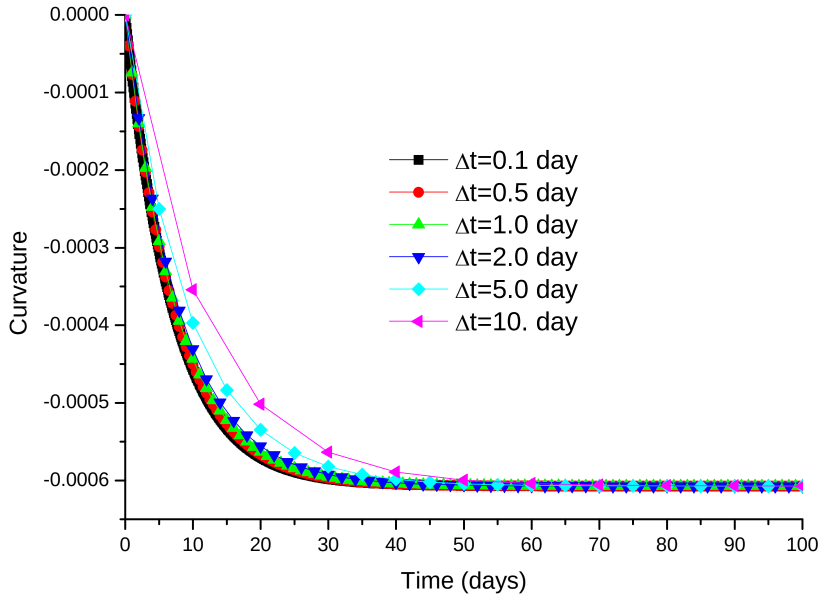

Figure 8 – Curvature at point 2.

As one can see the results present small dependence on time step length and are very stable.

3.2 Boltzmann model

In order to achieve the BEM Boltzmann viscoelastic formulation without internal cells and using a static fundamental solution it is necessary to apply the constitutive relation (25) in the right part of Bettis theorem, equation (26). It is also necessary to apply the elastic strain ε_{ij}^{e} for the left terms of the same equation, these actions result in:

$$\int_V \sigma_{ij}^{*}\varepsilon_{ij}^{e}dV = \int_V \left(\frac{E_e E_{ve}}{E_e + E_{ve}} C_{ij}^{lm}\left(\varepsilon_{lm} + \gamma\dot{\varepsilon}_{lm}\right) - \frac{\gamma E_{ve}}{E_e + E_{ve}}\dot{\sigma}_{ij} \right)\varepsilon_{ij}^{*}dV \tag{62}$$

or

$$\begin{aligned} \int_V \sigma_{ij}^{*}\varepsilon_{ij}^{e}dV = & \int_V \frac{E_e E_{ve}}{E_e + E_{ve}} C_{ij}^{lm}\varepsilon_{lm}\varepsilon_{ij}^{*}dV + \\ & + \int_V \frac{E_e E_{ve}}{E_e + E_{ve}} C_{ij}^{lm}\gamma\dot{\varepsilon}_{lm}\varepsilon_{ij}^{*}dV - \int_V \frac{\gamma E_{ve}}{E_e + E_{ve}}\dot{\sigma}_{ij}\varepsilon_{ij}^{*}dV \end{aligned} \tag{63}$$

Substituting (28) and (29) into (63) and integrating along the thickness one finds:

$$-\int_{\Omega} M_{ij}^{*} w,_{ij}^{e} \, d\Omega = -\int_{\Omega} \frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} w,_{lm} \, w,_{ij}^{*} \, d\Omega + \\ -\int_{\Omega} \frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} \gamma \dot{w},_{lm} \, w,_{ij}^{*} \, d\Omega + \int_{\Omega} \frac{\gamma D_{ve}}{D_e + D_{ve}} \dot{M}_{ij} w,_{ij}^{*} \, d\Omega \tag{64}$$

Writing the fundamental moments in terms of fundamental curvatures results:

$$-\int_{\Omega} D_e C_{ij}^{lm} w,_{lm}^{*} \, w,_{ij}^{e} \, d\Omega = -\int_{\Omega} \frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} w,_{lm} \, w,_{ij}^{*} \, d\Omega + \\ -\int_{\Omega} \frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} \gamma \dot{w},_{lm} \, w,_{ij}^{*} \, d\Omega + \int_{\Omega} \frac{\gamma D_{ve}}{D_e + D_{ve}} \dot{M}_{ij} w,_{ij}^{*} \, d\Omega \tag{65}$$

Observing that:

$$D_e C_{ij}^{lm} w,_{lm}^{*} \, w,_{ij}^{e} = D_e C_{ij}^{lm} w,_{ij}^{*} \, w,_{lm}^{e} = M_{ij}^{e} w,_{ij}^{*} = M_{ij} w,_{ij}^{*} \tag{66}$$

$$D_e C_{ij}^{lm} w,_{lm} \, w,_{ij}^{*} = D_e C_{ij}^{lm} w,_{ij} \, w,_{lm}^{*} = M_{ij}^{*} w,_{ij} \tag{67}$$

Equation (65) is rewritten as:

$$-\int_{\Omega} M_{ij} w,_{ij}^{*} \, d\Omega = -\int_{\Omega} \frac{D_{ev}}{D_e + D_{ve}} M_{ij}^{*} w,_{ij} \, d\Omega + \\ -\int_{\Omega} \gamma \frac{D_{ve}}{D_e + D_{ve}} M_{ij}^{*} \dot{w},_{ij} \, d\Omega + \int_{\Omega} \frac{\gamma D_{ve}}{D_e + D_{ve}} \dot{M}_{ij} w,_{ij}^{*} \, d\Omega \tag{68}$$

Reorganizing equation (68), one finds:

$$-\int_{\Omega} M_{ij}^{*} w,_{ij} \, d\Omega - \gamma \int_{\Omega} M_{ij}^{*} \dot{w},_{ij} \, d\Omega = -\frac{D_e + D_{ve}}{D_{ve}} \int_{\Omega} M_{ij} w,_{ij}^{*} \, d\Omega - \gamma \int_{\Omega} \dot{M}_{ij} w,_{ij}^{*} \, d\Omega \tag{69}$$

Using the Kirchhoff plate theory on equation (69) results the displacement integral equation for internal points q without domain integrals for the Boltzmann viscoelastic model, as:

$$\begin{aligned} & w(q) + \gamma \dot{w}(q) + \int_{\Gamma} \left(V_n^{*}(q,P) w(P) - M_n^{*}(q,P) \frac{\partial w}{\partial n}(P) \right) d\Gamma(P) + \\ & \gamma \int_{\Gamma} \left(V_n^{*}(q,P) \dot{w}(P) - M_n^{*}(q,P) \frac{\partial \dot{w}}{\partial n}(P) \right) d\Gamma(P) + \sum_{i=1}^{N_c} R_{ci}^{*}(q,P) w_{ci}(P) + \\ & + \gamma \sum_{i=1}^{N_c} R_{ci}^{*}(q,P) \dot{w}_{ci}(P) = \frac{D_e + D_{ve}}{D_{ve}} \int_{\Gamma} \left(V_n(P) w^{*}(q,P) - M_n(P) \frac{\partial w^{*}}{\partial n}(q,P) \right) d\Gamma(P) + \\ & \gamma \int_{\Gamma} \left(\dot{V}_n(P) w^{*}(q,P) - \dot{M}_n(P) \frac{\partial w^{*}}{\partial n}(q,P) \right) d\Gamma(P) + \frac{D_e + D_{ve}}{D_{ve}} \sum_{i=1}^{N_c} R_{ci}(P) w_{ci}^{*}(q,P) + \\ & + \gamma \sum_{i=1}^{N_c} \dot{R}_{ci}(P) w_{ci}^{*}(q,P) + \frac{D_e + D_{ve}}{D_{ve}} \int_{\Omega_g} g(p) w^{*}(q,p) d\Omega_g(p) + \\ & + \gamma \int_{\Omega_g} \dot{g}(p) w^{*}(q,p) d\Omega_g(p) \end{aligned} \tag{70}$$

For a boundary point Q one achieves:

$$
\begin{aligned}
&K(Q)w(Q)+\gamma K(Q)\dot{w}(Q)+\int_{\Gamma}\left(V_n^*(Q,P)w(P)-M_n^*(Q,P)\frac{\partial w}{\partial n}(P)\right)d\Gamma(P)+\\
&\gamma\int_{\Gamma}\left(V_n^*(Q,P)\dot{w}(P)-M_n^*(Q,P)\frac{\partial \dot{w}}{\partial n}(P)\right)d\Gamma(P)+\sum_{i=1}^{N_c}R_{ci}^*(Q,P)w_{ci}(P)+\\
&+\gamma\sum_{i=1}^{N_c}R_{ci}^*(Q,P)\dot{w}_{ci}(P)=\frac{D_e+D_{ve}}{D_{ve}}\int_{\Gamma}\left(V_n(P)w^*(Q,P)-M_n(P)\frac{\partial w^*}{\partial n}(Q,P)\right)d\Gamma(P)+\\
&\gamma\int_{\Gamma}\left(\dot{V}_n(P)w^*(Q,P)-\dot{M}_n(P)\frac{\partial w^*}{\partial n}(Q,P)\right)d\Gamma(P)+\frac{D_e+D_{ve}}{D_{ve}}\sum_{i=1}^{N_c}R_{ci}(P)w_{ci}^*(Q,P)+\\
&+\gamma\sum_{i=1}^{N_c}\dot{R}_{ci}(P)w_{ci}^*(Q,P)+\frac{D_e+D_{ve}}{D_{ve}}\int_{\Omega_g}g(p)w^*(Q,p)d\Omega_g(p)+\\
&+\gamma\int_{\Omega_g}\dot{g}(p)w^*(Q,p)d\Omega_g(p)
\end{aligned}
\tag{71}
$$

It is interesting to note that the differences between equations (37) and (70) and (38) and (71) are the coefficient evolving De and Dve, and the presence of γ multiplying some terms at the right side of equations (70) and (71). In order to find efforts at internal points, firstly one makes the derivatives of equation (70) to find curvatures, as:

$$
\begin{aligned}
&\frac{\partial^2 w(q)}{\partial x_k\partial x_l}+\gamma\frac{\partial^2 \dot{w}(q)}{\partial x_k\partial x_l}+\int_{\Gamma}\left(\frac{\partial^2 V_n^*}{\partial x_k\partial x_l}(q,P)w(P)-\frac{\partial^2 M_n^*}{\partial x_k\partial x_l}(q,P)\frac{\partial w}{\partial n}(P)\right)d\Gamma(P)+\\
&\gamma\int_{\Gamma}\left(\frac{\partial^2 V_n^*}{\partial x_k\partial x_l}(q,P)\dot{w}(P)-\frac{\partial^2 M_n^*}{\partial x_k\partial x_l}(q,P)\frac{\partial \dot{w}}{\partial n}(P)\right)d\Gamma(P)+\sum_{i=1}^{N_c}\frac{\partial^2 R_{ci}^*}{\partial x_k\partial x_l}(q,P)w_{ci}(P)+\\
&+\gamma\sum_{i=1}^{N_c}\frac{\partial^2 R_{ci}^*}{\partial x_k\partial x_l}(q,P)\dot{w}_{ci}(P)=\frac{D_e+D_{ve}}{D_{ve}}\int_{\Gamma}\left(V_n(P)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,P)+\right.\\
&\left.-M_n(P)\frac{\partial^2}{\partial x_k\partial x_l}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)d\Gamma(P)+\gamma\int_{\Gamma}\left(\dot{V}_n(P)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,P)+\right.\\
&\left.-\dot{M}_n(P)\frac{\partial^2}{\partial x_k\partial x_l}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)d\Gamma(P)+\frac{D_e+D_{ve}}{D_{ve}}\sum_{i=1}^{N_c}R_{ci}(P)\frac{\partial^2 w_{ci}^*}{\partial x_k\partial x_l}(q,P)+\\
&+\gamma\sum_{i=1}^{N_c}\dot{R}_{ci}(P)\frac{\partial^2 w_{ci}^*}{\partial x_k\partial x_l}(q,P)+\frac{D_e+D_{ve}}{D_{ve}}\int_{\Omega_g}g(p)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,p)d\Omega_g(p)+\\
&+\gamma\int_{\Omega_g}\dot{g}(p)\frac{\partial^2 w^*}{\partial x_k\partial x_l}(q,p)d\Omega_g(p)
\end{aligned}
\tag{72}
$$

$$\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w(q)}{\partial x_k \partial x_l}\right)+\gamma\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 \dot{w}(q)}{\partial x_k \partial x_l}\right)+\int_\Gamma\left(\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 V_n^*}{\partial x_k \partial x_l}(q,P)\right)w(P)+\right.$$
$$-\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 M_n^*}{\partial x_k \partial x_l}(q,P)\right)\frac{\partial w}{\partial n}(P)\Bigg)d\Gamma(P)+\gamma\int_\Gamma\left(\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 V_n^*}{\partial x_k \partial x_l}(q,P)\right)\dot{w}(P)+\right.$$
$$-\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 M_n^*}{\partial x_k \partial x_l}(q,P)\right)\frac{\partial \dot{w}}{\partial n}(P)\Bigg)d\Gamma(P)+\sum_{i=1}^{N_c}\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 R_{ci}^*}{\partial x_k \partial x_l}(q,P)\right)w_{ci}(P)+$$
$$+\gamma\sum_{i=1}^{N_c}\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 R_{ci}^*}{\partial x_k \partial x_l}(q,P)\right)\dot{w}_{ci}(P)=\frac{D_e+D_{ve}}{D_{ve}}\int_\Gamma\left(V_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_k \partial x_l}(q,P)\right)+\right.$$
$$-M_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)\Bigg)d\Gamma(P)+\gamma\int_\Gamma\left(\dot{V}_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_k \partial x_l}(q,P)\right)+\right.$$
$$-\dot{M}_n(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{\partial w^*}{\partial n}(q,P)\right)\right)\Bigg)d\Gamma(P)+\frac{D_e+D_{ve}}{D_{ve}}\sum_{i=1}^{N_c}R_{ci}(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w_{ci}^*}{\partial x_k \partial x_l}(q,P)\right)+$$
$$+\gamma\sum_{i=1}^{N_c}\dot{R}_{ci}(P)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w_{ci}^*}{\partial x_k \partial x_l}(q,P)\right)+\frac{D_e+D_{ve}}{D_{ve}}\int_{\Omega_g}g(p)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_k \partial x_l}(q,p)\right)d\Omega_g(p)+$$
$$+\gamma\int_{\Omega_g}\dot{g}(p)\frac{\partial}{\partial x_\beta}\left(\frac{\partial^2 w^*}{\partial x_k \partial x_l}(q,p)\right)d\Omega_g(p) \qquad (73)$$

From the above considerations and using standard boundary approximations one achieves the following system of time differential equations for the Boltzmann vicoelastic representation as:

$$K(Q)u(Q)+\gamma K(Q)\dot{u}(Q)+\bar{H}(Q)\bar{U}+\gamma\bar{H}(Q)\dot{\bar{U}}+\bar{H}_c(Q)\bar{w}_c+$$
$$+\gamma\bar{H}_c(Q)\dot{\bar{w}}_c=\frac{D_e+D_{ve}}{D_{ve}}\bar{G}(Q)\bar{P}+\gamma\bar{G}(Q)\dot{\bar{P}}+\frac{D_e+D_{ve}}{D_{ve}}\bar{G}_c(Q)\bar{R}_c+ \qquad (74)$$
$$\gamma\bar{G}_c(Q)\dot{\bar{R}}_c+\frac{D_e+D_{ve}}{D_{ve}}\bar{T}(Q)+\gamma\dot{\bar{T}}(Q)$$

for which there are additional terms when compared to the Kelvin-Voigt model, i.e., $\dot{\bar{\mathrm{P}}}$ is the nodal time rate of boundary reactions, given by:

$$\dot{\bar{\mathrm{P}}}=\left\{\dot{\mathrm{V}}_n^1 \quad \dot{\mathrm{M}}_n^1 \quad \dot{\mathrm{V}}_n^2 \quad \dot{\mathrm{M}}_n^2 \quad \dots \quad \dot{\mathrm{V}}_n^{N_n} \quad \dot{\mathrm{M}}_n^{N_n}\right\}$$

$\dot{\bar{\mathrm{R}}}_c$ is the time rate of corner reactions, i.e.:

$$\dot{\bar{\mathrm{R}}}_c=\left\{\dot{\mathrm{R}}_c^1 \quad \dot{\mathrm{R}}_c^2 \quad \dots \quad \dot{\mathrm{R}}_c^{N_c}\right\}$$

And $\dot{\mathrm{T}}(\mathrm{Q})$ is the time rate of loading over region Ωg,.

Introducing, as made before, $\mathrm{K}(\mathrm{Q})$ into $\bar{\mathrm{H}}(\mathrm{Q})$, considering $\bar{\mathrm{R}}_c$ zero and $\bar{\mathrm{w}}_c$ a function of neighbour points, results:

$$\overline{H}\overline{U} + \gamma \overline{H}\dot{\overline{U}} = \frac{D_e + D_{ve}}{D_{ve}} \overline{G}\overline{P} + \gamma \overline{G}\dot{\overline{P}} + \frac{D_e + D_{ve}}{D_{ve}} \overline{T} + \gamma \dot{\overline{T}} \tag{75}$$

Following the same reasoning, the internal points displacements, curvatures and the derivatives of curvatures, given by equations (70), (72) and (73), respectively, can be written as::

$$\overline{u}(q) + \gamma \dot{\overline{u}}(q) + \overline{H}'\overline{U} + \gamma \overline{H}'\dot{\overline{U}} = \frac{D_e + D_{ve}}{D_{ve}} \overline{G}'\overline{P} + \gamma \overline{G}'\dot{\overline{P}} + \frac{D_e + D_{ve}}{D_{ve}} \overline{T}' + \gamma \dot{\overline{T}}' \tag{76}$$

Time integration should be done in order to solve equation (75) along time. As the time derivatives are of the first order with constant coefficients it is enough to adopt a simple linear approximation for all time derivatives over a time step as:

$$\dot{w} = \frac{w_{(s+1)} - w_{(s)}}{\Delta t} \tag{77}$$

$$\frac{\partial \dot{w}}{\partial n} = \frac{\left.\dfrac{\partial w}{\partial n}\right|_{(s+1)} - \left.\dfrac{\partial w}{\partial n}\right|_{(s)}}{\Delta t} \tag{78}$$

$$\dot{V}_n = \frac{V_{n(s+1)} - V_{n(s)}}{\Delta t} \tag{79}$$

$$\dot{M}_n = \frac{M_{n(s+1)} - M_{n(s)}}{\Delta t} \tag{80}$$

$$\dot{g} = \frac{g_{(s+1)} - g_{(s)}}{\Delta t} \tag{81}$$

Using the vector representation of $\dot{\overline{\mathrm{U}}}$, $\dot{\overline{\mathrm{P}}}$ and $\dot{\overline{\mathrm{T}}}$ one writes equations (77) through (81) in a unified version, as:

$$\dot{\overline{U}} = \frac{\overline{U}_{(s+1)} - \overline{U}_{(s)}}{\Delta t} \tag{82}$$

$$\dot{\overline{P}} = \frac{\overline{P}_{(s+1)} - \overline{P}_{(s)}}{\Delta t} \tag{83}$$

$$\dot{\overline{T}} = \frac{\overline{T}_{(s+1)} - \overline{T}_{(s)}}{\Delta t} \tag{84}$$

Substuting (82) through (84) into (75), results:

$$\tilde{H}\overline{U}_{(s+1)} = \tilde{G}\overline{P}_{(s+1)} + \tilde{T} + \tilde{F}_s \tag{85}$$

where:

$$\tilde{H} = \left(1 + \frac{\gamma}{\Delta t}\right)\overline{H} \tag{86}$$

$$\tilde{G} = \left(\frac{D_e + D_{ve}}{D_{ve}} + \frac{\gamma}{\Delta t}\right)\overline{G} \tag{87}$$

$$\tilde{T} = \left(\frac{D_e + D_{ve}}{D_{ve}} + \frac{\gamma}{\Delta t}\right)\overline{T}_{(s+1)} \tag{88}$$

$$\tilde{F}_s = \frac{\gamma}{\Delta t}\overline{H}\overline{U}_{(s)} - \frac{\gamma}{\Delta t}\overline{G}\overline{P}_{(s)} - \frac{\gamma}{\Delta t}\overline{T}_{(s)} \tag{89}$$

To impose boundary conditions one changes columns of $\tilde{\mathrm{H}}$ e $\tilde{\mathrm{G}}$. As usual, the right side of the resulting equation has only known values that can be put together and the system (85) can be solved achieving the current displacement and reactions. From these values the time rates of all variables are calculated by equations (82), (83) and (84).

To calculate current internal values, i.e., displacements, curvatures, and derivatives of curvatures, equation (76) is adopted using time approximation (77) and the following:

$$\dot{w},_{ij} = \frac{w,_{ij(s+1)} + w,_{ij(s)}}{\Delta t} \tag{90}$$

$$\dot{w},_{kk\beta} = \frac{w,_{kk\beta(s+1)} + w,_{kk\beta(s)}}{\Delta t} \tag{91}$$

That is,

$$\dot{\overline{u}} = \frac{\overline{u}_{(s+1)} - \overline{u}_{(s)}}{\Delta t} \tag{92}$$

Applying (92) into (76) results:

$$\overline{u}_{(s+1)} = \left(-\overline{H}'\overline{U} - \gamma\overline{H}'\dot{\overline{U}} + \frac{D_e + D_{ve}}{D_{ve}}\overline{G}'\overline{P} + \gamma\overline{G}'\dot{\overline{P}} + \right. \\ \left. + \frac{D_e + D_{ve}}{D_{ve}}\overline{T}' + \gamma\dot{\overline{T}}' + \frac{\gamma}{\Delta t}\overline{u}_{(s)} \right) \Bigg/ \left(1 + \frac{\gamma}{\Delta t}\right) \tag{93}$$

From the result of equation (93) one calculates the time rates for internal displacements, curvatures and derivatives of curvatures using equations (77), (90) and (91). To calculate total moments one should use what follows:

$$\dot{M}_{ij} = \frac{M_{ij(s+1)} - M_{ij(s)}}{\Delta t} \tag{94}$$

Rewriting equation (25) for plates, results:

$$M_{ij} = \frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} \left(w,_{lm} + \gamma \dot{w},_{lm} \right) - \frac{\gamma D_{ve}}{D_e + D_{ve}} \dot{M}_{ij} \tag{95}$$

Substituting (94) into (95), one finds:

$$M_{ij(s+1)} = \left(\frac{D_e D_{ve}}{D_e + D_{ve}} C_{ij}^{lm} \left(w,_{lm} + \gamma \dot{w},_{lm} \right) + \frac{D_{ve}}{D_e + D_{ve}} \frac{\gamma}{\Delta t} M_{ij(s)} \right) \Big/ \left(1 + \frac{D_{ve}}{D_e + D_{ve}} \frac{\gamma}{\Delta t} \right) \tag{96}$$

In order to calculate the viscous and elastic parts of the total moment it is necessary to apply a relation developed by MESQUITA & CODA (2001) that consists for plate analysis in what follows.

Deriving equation (15) regarding time one writes:

$$\dot{M}_{ij}^{el} = D_{ve} C_{ij}^{lm} \dot{w},_{lm}^{ve} = \frac{1}{\gamma} \gamma D_{ve} C_{ij}^{lm} \dot{w},_{lm}^{ve} = \frac{1}{\gamma} M_{ij}^{v} \tag{97}$$

or

$$M_{ij}^{v} = \gamma \dot{M}_{ij}^{el} \tag{98}$$

Rewriting equation (20) for plates results:

$$M_{ij} = M_{ij}^{el} + M_{ij}^{v} \tag{99}$$

Applying (98) into (99), results the following differential equation:

$$\gamma \dot{M}_{ij}^{el} + M_{ij}^{el} - M_{ij} = 0 \tag{100}$$

This differential equation is solved adopting the following approximation:

$$\dot{M}_{ij}^{el} = \frac{M_{ij(s+1)}^{el} - M_{ij(s)}^{el}}{\Delta t} \tag{101}$$

Substituting (101) into (100), it follows:

$$M_{ij(s+1)}^{el} = \left(M_{ij(s+1)} + \frac{\gamma}{\Delta t} M_{ij(s)}^{el} \right) \Big/ \left(1 + \frac{\gamma}{\Delta t} \right) \tag{102}$$

From this equation one finds the elastic moment from the total moment and using equation (99) one recovers the viscous moment.

To find the total shear force one substitutes

$$M_{ij},_i - q_j = 0 \tag{103}$$

Into equation (95) resulting

$$q_\beta = -\frac{D_e D_{ve}}{D_e + D_{ve}}\left(w,_{kk\beta} + \gamma \dot{w},_{kk\beta}\right) - \frac{\gamma D_{ve}}{D_e + D_{ve}}\dot{q}_\beta \tag{104}$$

Using the following approximation

$$\dot{q}_\beta = \frac{q_{\beta(s+1)} - q_{\beta(s)}}{\Delta t} \tag{105}$$

into equation (104), results

$$q_{\beta(s+1)} = \left(-\frac{D_e D_{ve}}{D_e + D_{ve}}\left(w,_{kk\beta} + \gamma \dot{w},_{kk\beta}\right) + \frac{D_{ve}}{D_e + D_{ve}}\frac{\gamma}{\Delta t} q_{\beta(s)}\right) \Big/ \left(1 + \frac{D_{ve}}{D_e + D_{ve}}\frac{\gamma}{\Delta t}\right) \tag{106}$$

One is able to calculate the elastic and viscous parts of shear force using the same reasoning applied for moments, i.e.:

$$q^{el}_{\beta(s+1)} = \left(q_{\beta(s+1)} + \frac{\gamma}{\Delta t} q^{el}_{\beta(s)}\right) \Big/ \left(1 + \frac{\gamma}{\Delta t}\right) \tag{107}$$

Completing the procedure

3.2.1 Numerical example for Boltzmann model

This example is similar to the one presented in item 2. Dimensions, discretization and applied load are the same depicted in figure 3. The physical parameters are the same except the existence of an instantaneous elastic modulus of $\mathrm{E_{el}} = 1.25\mathrm{x}10^7\,\mathrm{kN/m^2}$ and other to the viscoelastic part given by $E_{ve} = 2,5x10^7\,kN/m^2$.

The results for curvatures w,11 and w,22 along time for point A are presented in figure 9 and for moments M22 are presented in figure 10.

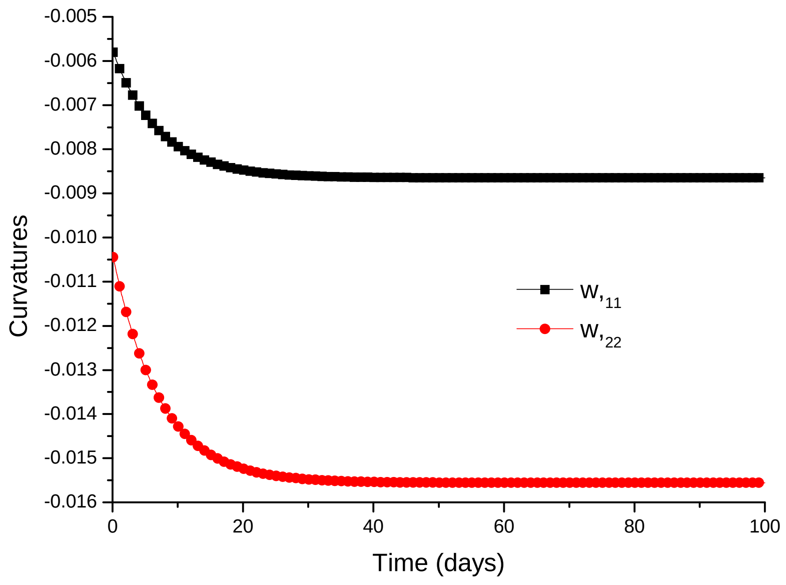

Figure 9 – Curvatures for point A.

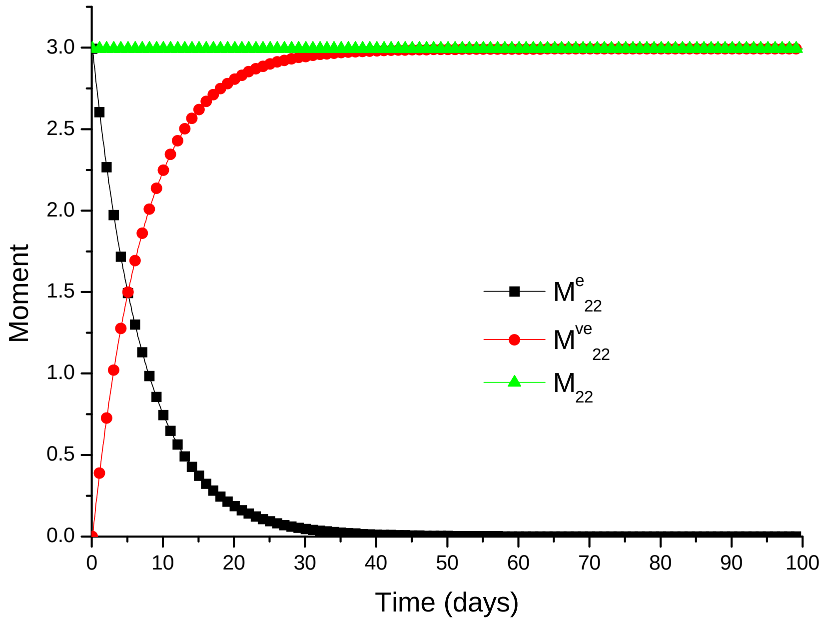

Figure 10 – Moments M22 for point A.

Figure 9 shows the instantaneous and viscous structural behavior due to the Boltzmann model. Figure 10 shows the transfer of efforts from the viscous and elastic parts of the viscoelastic fragment of the Boltzmann model. The moment along the elastic (instantaneous) fragment of the model is equal to the total moment. After 50 days the results are practically constant along time.

3.2.2 General plate

This example uses the proposed viscoelastic formulations for a more elaborated problem. It is a plate in bending with two sides partially supported, one side totally supported and the last one free. A uniformly distributed load is applied as depicted in figure 11.

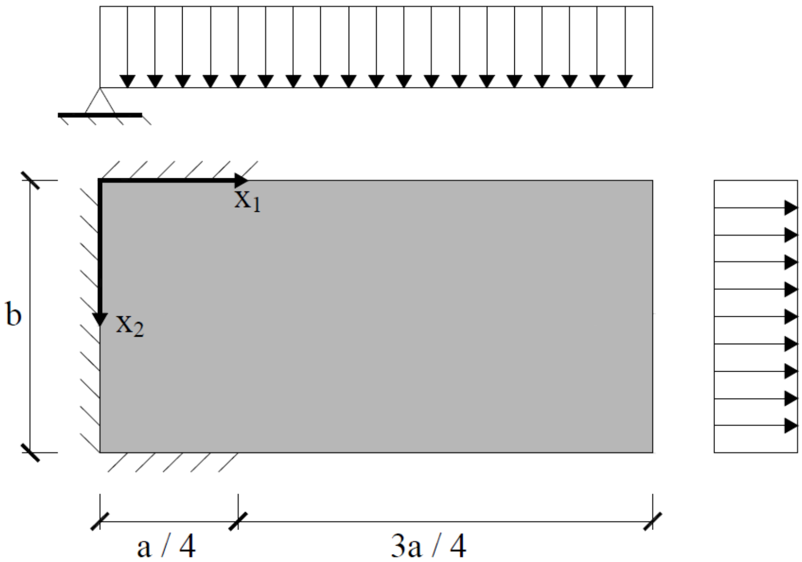

Figure 11 – Geometry and loading.

The adopted material properties for the Kelvin-Voigt model are $E=2.5\,x10^7\,kN/m^2$, $\nu=0.3$ and $\gamma=7.14285\,days$. For the Boltzmann model the physical properties are: $E_e=1.5\,x10^7\,kN/m^2$, $E_{ve}=2.5\,x10^7\,kN/m^2$, $\nu=0.3$ and $\gamma=7.14285\,days$. The load is instantly applied and its value is $q=10kN/m^2$. The adopted geometric parameters are: $a=2m$, $b=1m$ and $t=0.06m$. The adopted time step is $\Delta t=0.1days$ for a total of $100\ days$. As the good behavior of the formulations regarding time has been presented in the previous examples, to save space, only the final result is shown, i.e., at day 100. Two discretizations have been adopted, the one called A has 24 elements and 54 nodes while discretization B has 192 elements and 390 nodes. Displacements for Kelvin-Voigt model (figure 12) and reactions for Boltzmann model (figure 13) at the partially supported side of the plate are analyzed for both discretizations.

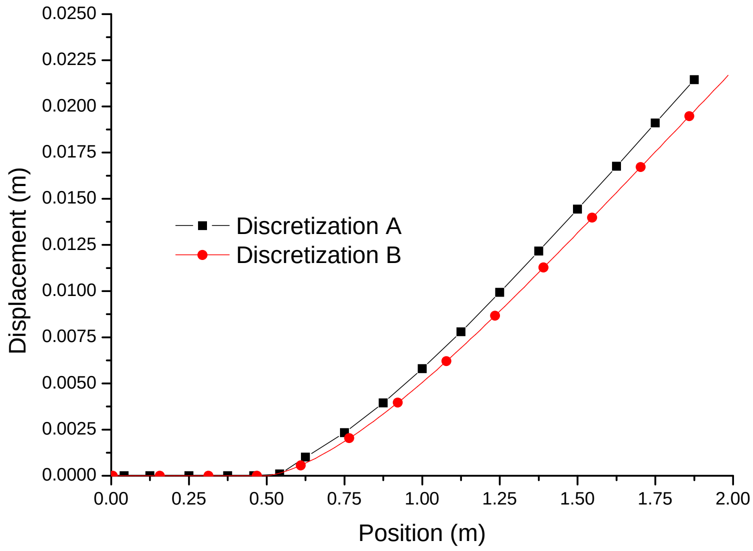

Figure 12 – Displacement for Kelvin model (100 days).

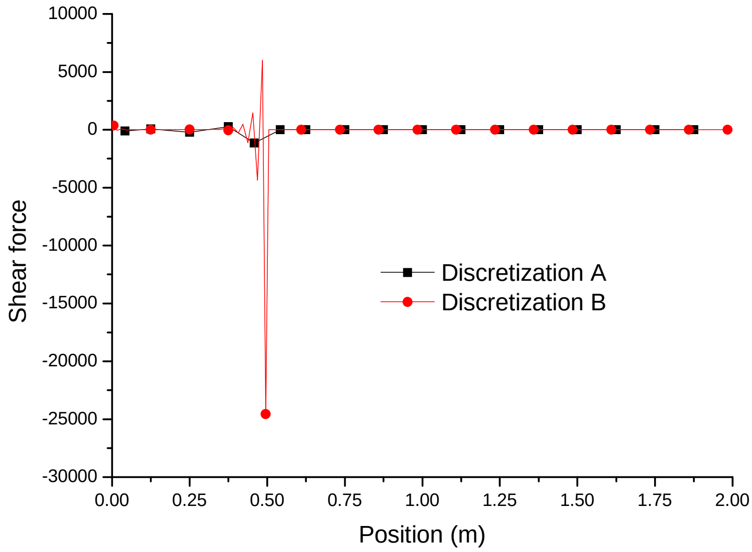

Figure 13 – Reaction for longitudinal faces full range.

4 CONCLUSIONS

The presented formulations are important contributions for the BEM research as the authors did not find any analytical or numerical solution to compare results including the viscous behaviors of plates by the BEM. The adaptation of the differential constitutive model proposed by Mesquita et al. (2004) for plate analysis by the BEM has been successfully carried out. The resulting computational code is accurate and present a small computational effort when compared with other ways to solve this kind of problems, e.g., Laplace transforms or convolution solutions.

The presented formulations, Kelvin-Voigt and Boltzmann are efficient and elegant, as no discretizations over the domain where required. One interesting feature is the total flexibility to apply variable loads along time for both models. The authors indicate further investigations and improvements in order to identify the origin of the vertical reaction oscillations present in partially supported sides.

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