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DYNAMIC ANALYSIS OF ELASTIC-PLASTIC SATURATED POROUS MEDIA BY THE BOUNDARY ELEMENT METHOD

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Abstract

This paper presents a time-domain boundary element formulation for the dynamic analysis of saturated porous media. Integral equations for displacements, stresses and pore-pressures, based on non-transient fundamental solutions are considered. Elastoplastic models are also dealt with by the present methodology, extending the applicability of boundary elements to model complex pore-dynamic problems. At the end of the paper, a discussion concerning two numerical examples is presented, illustrating the potentialities of the new procedure.

Keywords: Boundary elements. Time-domain. Porous media. Elastoplasticity.

1 INTRODUCTION

For many everyday engineering problems, such as earthquake engineering, soil-structure interaction, biomechanics, seismic wave scattering etc., dynamic porous media analysis is necessary and over simplified theoretical models (e.g., pure elastic theory etc.) may only represent a very crude approximation. The pioneering work of Biot (1941, 1956a, 1956b, 1962) is commonly referred to as the beginning of an era of study of porodynamic problems. For a quite complete overview of the porous media theory evolution (which is marked with very peculiar and interesting aspects) the work of de Boer (1998) is recommended.

In the early years of development, classical mathematics was the only effective tool available to solve the set of governing differential equations describing porous models; thereby, the complexity of the analyses was quite limited. In the last decades, with the drastic evolution of digital computers, numerical methods (mainly FDM, FEM and BEM) have assumed an important role in the solution of practical and complex engineering problems. However, for the dynamic analysis of porous media, the Boundary Element Method (BEM) application is still limited to "simple" case analyses (linear models etc.). Moreover, the current time-domain based boundary element formulations are highly CPU-time demanding. In fact, most of the BEM limitations concerning time-domain pore-dynamic analyses are due to the lack of an appropriate time-dependent fundamental solution. Unfortunately, this still remains an open task.

One of the earliest books on porous media, taking into account boundary elements, was presented by Liggett and Liu (1983). Concerning poroelastodynamic analyses, Predeleanu (1984) and Manolis and Beskos (1989) are among the first to introduce boundary element formulation procedures. The fundamental solution employed was based on transformed domains, namely the Laplace domain (Manolis and Beskos, 1989) and the frequency domain (Norris, 1985). However, these formulations were based on solid and fluid displacements (six unknowns) and it can be shown (Bonnet, 1987; Boutin et al., 1987) that only solid displacements and fluid pressures are independent (four unknowns). Boundary element formulations based on these four independent variables (in fact, three independent variables since only plane strain problems were focused) were presented by Cheng et al. (1991) and Domínguez (1992), in a frequency domain context.

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Due to the lack of time-dependent Green's functions, time-domain boundary element formulations for porous media have been developed mostly based on transformed-domain fundamental solutions. Works in this context have been presented by Chen and Dargush (1995), which uses analytical inverse transformation of Laplace domain fundamental solutions (Chen, 1994a, 1994b), and Schanz (2001a, 2001b), which employs the convolution quadrature method, proposed by Lubich (1988a, 1988b). A formulation based on a time-domain fundamental solution was developed by Wiebe and Antes (1991), neglecting the viscous coupling of Biot's dynamic poroelasticity (vanishing damping between the solid skeleton and the fluid).

Considering nonlinear porous media modelled by the BEM, consolidation analyses were presented by Benallal et al. (2008) (see also Cavalcanti and Telles, 2003, Venturini et al., 2005a-c, etc. for other references considering consolidation analyses by the BEM), whereas the dynamic analysis was introduced by Soares Jr. et al. (2006).

Time-domain boundary element formulations based on transformed-domain fundamental solutions are CPU-time demanding and quite elaborated mathematically. Thus, the present paper presents a time-domain BEM analysis based on non-transient fundamental solutions (2D form of the Kelvin solution). The equations that arise are mathematically simpler to deal with, making it possible to consider more complex physical phenomena without that much effort (e.g., elastoplasticity). Moreover, time-marching schemes related to this kind of analysis (see the Houbolt method, for instance; Houbolt, 1950) present low computational costs. The drawback of the methodology is that it keeps domain integrals in the formulation (inertial terms, coupling terms etc.). The necessity of domain discretization, however, does not turn the formulation unattractive. As pointed out by Telles (1983) and Venturini (1984), when performing elastoplastic analyses, the part of the domain where inelastic behaviour is expected to occur necessarily requires domain discretization. Thus, as future developments, the present methodology (modelling, for instance, non-linear complex sub-domains) may be coupled with other time-domain boundary element formulations (e.g., Mansur, 1883, Coda and Venturini, 1995, etc.). Thereby, in an infinite domain context, the interface between the different methodologies could be interpreted as an efficient non-reflecting boundary. For details on this coupling of different BE formulations concerning dynamic analyses, the work of Soares Jr et al. (2005) is recommended.

In the present paper, firstly (section 2) the governing equations of the problem are presented and briefly discussed. A complete set of equations is initially considered and, later on, simplifications on the general formulation are adopted, pointing out the main expressions to be worked out here (*u-p formulation*). In section 3, boundary element procedures for the dynamic analysis of non-linear porous media are developed. Finally, at the end of the paper (section 4), two numerical applications are discussed, illustrating the potentialities of the new methodology. In appendix A, the employed non-transient fundamental solutions are presented.

2 GOVERNING EQUATIONS

For a unit volume and from the definition of the total stress, the total momentum equilibrium equation for the solid-fluid ensemble can be written as (Zienkiewicz et al., 1990)

$$\sigma_{ij,j} - \rho_m \ddot{u}_i + \rho_m b_i = \rho_f \left(\dot{w}_i + w_j w_{i,j}\right) \tag{1}$$

where σ_{ij} is the total Cauchy stress, using the usual indicial notation for Cartesian axes; the effective stress is defined as $\sigma'_{ij} = \sigma_{ij} + \alpha\delta_{ij} p$, in which α is the so-called Biot's parameter, accounting for slight strain changes, and p is the pore pressure. In addition, u_i stands for the solid matrix displacement, w_i for the mean fluid velocity relative to the solid phase and b_i for the body force distribution. Inferior commas and overdots indicate partial space ($u_{j,i} = \partial u_j / \partial x_i$) and time ($\dot{u}_i = \partial u_i / \partial t$) derivatives, respectively. The density of the mixture is defined as $\rho_m = \mu\rho_f + (1-\mu)\rho_s$, where ρ_s and ρ_f are the density of the solid and fluid phase, respectively, and μ is the porosity of the solid.

The constitutive law can be written, incrementally, as

$$d\sigma'_{ij} = D_{ijkl}\ (d\varepsilon_{kl} - d\varepsilon^0_{kl}) + \sigma'_{ik}\ d\omega_{kj} + \sigma'_{jk}\ d\omega_{ki} \tag{2}$$

where the last two terms account for the Zaremba-Jaumann rotational stress changes (negligible generally in small displacement computation) and D_{ijkl} is a fourth order tangential tensor defined by suitable state variables and the direction of the increment. The incremental strain $d\varepsilon_{ij} = (1/2)(du_{i,j} + du_{j,i})$ and respective rotation $d\omega_{ij} = (1/2)(du_{j,i} - du_{i,j})$ components are defined in the usual way from incremental displacement derivatives and ε^0_{ij} refers to initial strains caused by external actions such as temperature changes, creep, etc.

For a unit control volume, assumed attached to the solid phase and moving with it, the momentum equilibrium equation for the fluid alone can be written as

$$-k\,p,_i - w_i + k\rho_f(b_i - \ddot{u}_i) = \rho_f\left(\dot{w}_i + w_j w_{i,j}\right)/\mu \tag{3}$$

where k is the isotropic permeability coefficient, according to D'Arcy's seepage law.

The equation of flow conservation for the fluid phase can be written in the following form

$$(1/Q)\dot{p} + \alpha\dot{\varepsilon}_{ii} + w_{i,i} + \dot{s}_0 = -(\rho_f/\rho_m)\dot{w}_i \tag{4}$$

where $(1/Q) = \mu/K_f + (\alpha-\mu)/K_s$ and the compression modules of the solid and fluid phases are represented by K_s and K_f, respectively. The rate of volume changes of the fluid is $\dot{s}_0$.

2.1 Simplified equations – the u-p formulation

When the acceleration spectrum is composed of low frequencies (i.e., high frequencies contributions can be disregarded), the right hand side of equations (1), (3) and (4) involving the relative acceleration of the fluid are not important and can be omitted with confidence (Zienkiewicz and Shiomi, 1984). The omission of such terms allows for w_i to be eliminated from the governing system of equations, retaining only u_i and p as primary variables. The simplified final system of equations, also considering the dynamic seepage forcing term (i.e., $k\rho_f \ddot{u}_i,_i$) as negligible, can be written as

$$\sigma_{ij,j} - \rho_m \ddot{u}_i + \rho_m b_i = 0 \tag{5}$$

$$\alpha \dot{\varepsilon}_{ii} - k\, p,_{ii} + (1/Q)\dot{p} + k\rho_f b_i,_i + \dot{s}_0 = 0 \tag{6}$$

Equations (5) and (6), accompanied by appropriate initial and boundary conditions, define the model to be solved by the boundary element formulation here presented. Physical nonlinearities are taken into account (elastoplastic models) within the context of small strain theory.

3 BOUNDARY ELEMENT SOLUTION

The integral equations, which represent equations (5)-(6) for displacements and stresses, taking into account non-transient fundamental solutions and initial stress contributions (i.e., plastic strains and pore-pressures), are given by

$$\begin{aligned} c_{ik}(\xi)\, u_k(\xi,t) = & \int_\Gamma u_{ik}*(X;\xi)\tau_k(X,t)d\Gamma(X) - \int_\Gamma \tau_{ik}*(X;\xi)u_k(X,t)d\Gamma(X) + \\ & - \int_\Omega u_{ik}*(X;\xi)\rho_m\{\ddot{u}_k(X,t) - b_k(X,t)\}d\Omega(X) + \\ & + \int_\Omega \varepsilon_{ikj}*(X;\xi)\{\sigma^P_{kj}(X,t) + \alpha\delta_{kj}\, p(X,t)\}d\Omega(X) \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma_{ik}(\xi,t) = & \int_\Gamma u_{ikj}*(X;\xi)\tau_j(X,t)d\Gamma(X) - \int_\Gamma \tau_{ikj}*(X;\xi)u_j(X,t)d\Gamma(X) + \\ & - \int_\Omega u_{ikj}*(X;\xi)\rho_m\{\ddot{u}_j(X,t) - b_j(X,t)\}d\Omega(X) + \\ & + \int_\Omega \varepsilon_{ikjl}*(X;\xi)\{\sigma^P_{jl}(X,t) + \alpha\delta_{jl}\, p(X,t)\}d\Omega(X) + \\ & + g_{ik}\left(\sigma^P_{jl}(X,t) + \alpha\delta_{jl}\, p(X,t)\right) \end{aligned} \tag{8}$$

Taking the space derivative of equation (3), substituting the term $w_{i,i}$ from equation (4) and considering the simplifications of section 2.1 yields $\nabla^2 p = z_1\dot{p} + z_2\dot{\sigma}^e_{vol} + S$ (see equation (6)), in which $z_1 = (1/Q)/k$, $z_2 = (\alpha K_{nD})/k$ and $S = \dot{s}_0 + k\rho_f b_{i,i}$. For plane strain problems $K_{2D} = (1-2\nu)/G$, where ν and G are the solid skeleton drained Poisson ratio and shear modulus, respectively. The following integral equation is therefore obtained

$$c(\xi)\ p(\xi,t) = \int_{\Gamma} p^*(X;\xi)q(X,t)d\Gamma(X) - \int_{\Gamma} q^*(X;\xi)p(X,t)d\Gamma(X) + \\ - \int_{\Omega} p^*(X;\xi)\, z_1 \dot{p}(X,t)d\Omega(X) - \int_{\Omega} p^*(X;\xi)\, z_2 \dot{\sigma}^e_{vol}(X,t)d\Omega(X) \\ + \int_{\Omega} p^*(X;\xi)S(X,t)d\Omega(X) \tag{9}$$

where $\sigma^P_{ij} = D_{ijkl}\, \varepsilon^p_{kl}$ is the "plastic" stress tensor and $\dot{\sigma}^e_{vol}$ ($\sigma^e_{ij} = D_{ijkl}\, \varepsilon_{kl}$) is the volumetric "elastic" stress. S stands for the domain forces related to equation (6). Also, τi and q are tractions and fluxes, respectively, along the boundary Γ. The fundamental solutions uik*,τik*, εikj*, uikj*,τikj*, εikjl*, p* and q*, included in equations (7), (8) and (9), are presented in appendix A (as well as the definition of g_{ik}).

In order to solve equations (7)-(9), boundary elements and integration cells are employed to discretize the boundary and domain of the model, respectively. Hence, polynomial functions $\eta^j(X)$ are used with time dependent nodal values, as indicated below

$$u_k(X,t) = \sum_{j=1}^{J} \eta^j_u(X)\, u_{kj}(t) \tag{10}$$

By substituting approximations of the type indicated by equation (10) into equations (7)-(9), the following system of equations can be defined in matrix form

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{U}^n \\ {}_d\mathbf{U}^n \end{bmatrix} = \begin{bmatrix} {}_{bb}\mathbf{G} & \mathbf{0} \\ {}_{db}\mathbf{G} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{T}^n \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} {}_{bb}\mathbf{H} & \mathbf{0} \\ {}_{db}\mathbf{H} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{U}^n \\ {}_d\mathbf{U}^n \end{bmatrix} + \\ - \begin{bmatrix} {}_{bb}\mathbf{M} & {}_{bd}\mathbf{M} \\ {}_{db}\mathbf{M} & {}_{dd}\mathbf{M} \end{bmatrix} \begin{bmatrix} {}_b\ddot{\mathbf{U}}^n \\ {}_d\ddot{\mathbf{U}}^n \end{bmatrix} + \begin{bmatrix} {}_{bb}\mathbf{W} & {}_{bd}\mathbf{W} \\ {}_{db}\mathbf{W} & {}_{dd}\mathbf{W} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{O}^n_p + {}_b(\mathbf{mP}^n) \\ {}_d\mathbf{O}^n_p + {}_d(\mathbf{mP}^n) \end{bmatrix} + \mathbf{S}^n \tag{11}$$

$$\begin{bmatrix} {}_b\mathbf{O}^n \\ {}_d\mathbf{O}^n \end{bmatrix} = \begin{bmatrix} {}_{bb}\mathbf{G'} & \mathbf{0} \\ {}_{db}\mathbf{G'} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{T}^n \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} {}_{bb}\mathbf{H'} & \mathbf{0} \\ {}_{db}\mathbf{H'} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{U}^n \\ {}_d\mathbf{U}^n \end{bmatrix} + \\ - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ {}_{db}\mathbf{M'} & {}_{dd}\mathbf{M'} \end{bmatrix} \begin{bmatrix} {}_b\ddot{\mathbf{U}}^n \\ {}_d\ddot{\mathbf{U}}^n \end{bmatrix} + \begin{bmatrix} {}_{bb}\mathbf{W'} & \mathbf{0} \\ {}_{db}\mathbf{W'} & ({}_{dd}\mathbf{W'} + {}_{dd}\hat{\mathbf{W}}') \end{bmatrix} \begin{bmatrix} {}_b\mathbf{O}^n_p + {}_b(\mathbf{mP}^n) \\ {}_d\mathbf{O}^n_p + {}_d(\mathbf{mP}^n) \end{bmatrix} + \mathbf{S'}^n \tag{12}$$

$$\begin{bmatrix} \mathbf{C''} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{P}^n \\ {}_d\mathbf{P}^n \end{bmatrix} = \begin{bmatrix} {}_{bb}\mathbf{G''} & \mathbf{0} \\ {}_{db}\mathbf{G''} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{Q}^n \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} {}_{bb}\mathbf{H''} & \mathbf{0} \\ {}_{db}\mathbf{H''} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}_b\mathbf{P}^n \\ {}_d\mathbf{P}^n \end{bmatrix} + \\ - \begin{bmatrix} {}_{bb}\mathbf{M''} & {}_{bd}\mathbf{M''} \\ {}_{db}\mathbf{M''} & {}_{dd}\mathbf{M''} \end{bmatrix} \begin{bmatrix} {}_b\dot{\mathbf{P}}^n \\ {}_d\dot{\mathbf{P}}^n \end{bmatrix} + \begin{bmatrix} {}_{bb}\mathbf{W''} & {}_{bd}\mathbf{W''} \\ {}_{db}\mathbf{W''} & {}_{dd}\mathbf{W''} \end{bmatrix} \begin{bmatrix} {}_b\dot{\mathbf{O}}^n_e \\ {}_d\dot{\mathbf{O}}^n_e \end{bmatrix} + \mathbf{S''}^n \tag{13}$$

where H, $\mathbf{H'}$, $\mathbf{H''}$, G, $\mathbf{G'}$, $\mathbf{G''}$ are influence matrices based on boundary integrals and M, $\mathbf{M'}$, $\mathbf{M''}$, W, $\mathbf{W'}$, $\mathbf{W''}$ are influence matrices based on domain integrals. C, $\mathbf{C''}$ and $\hat{\mathbf{W}}'$ are related to cik, c and g_{ik}, respectively. In addition, m stands for the Biot-Kronecker parameter αδij and U, P, T and Q are the displacement, pore-pressure, traction and flux vectors, respectively. O is the total stress vector, Oe and Op are the elastic and plastic (effective) stress vectors, respectively. Matrices S, $\mathbf{S'}$, $\mathbf{S''}$ are related to the domain forces in equations (7)-(9). In equations (11)-(13), the subscripts b and d denote, respectively, boundary and domain associated coefficients and unknowns (the computation of boundary stresses has been carried out using the locally interpolated element tractions and displacements; hence, the hypersingular boundary integral equation has been avoided). Also, I and 0 represent identity and null matrices, respectively, and n stands for the current time ($t_n = n\Delta t$, where Δt is the adopted time-step).

Taking into account the definition $\sigma'_{ij} = \sigma_{ij} + \alpha\delta_{ij}p$, equations (11)-(13) can be re-written, in a concise manner, as

$$\mathbf{CU}^n = \mathbf{GT}^n - \mathbf{HU}^n - \mathbf{M\ddot{U}}^n + \mathbf{W}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) + \mathbf{S}^n \tag{14}$$

$$\mathbf{O'}^n = \mathbf{G'T}^n - \mathbf{H'U}^n - \mathbf{M'\ddot{U}}^n + \mathbf{W'}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) + \mathbf{mP}^n + \mathbf{S'}^n \tag{15}$$

$$\mathbf{C''P}^n = \mathbf{G''Q}^n - \mathbf{H''P}^n - \mathbf{M''\dot{P}}^n + \mathbf{W''\dot{O}}_e^n + \mathbf{S''}^n \tag{16}$$

where $\mathbf{O'}$ is the effective stress vector.

In the present work, the time-dependent problem is solved taking into account the Houbolt method (Houbolt, 1950)

$$\ddot{\mathbf{V}}^n = (2\mathbf{V}^n - 5\mathbf{V}^{n-1} + 4\mathbf{V}^{n-2} - \mathbf{V}^{n-3}) / \Delta t^2 \tag{17}$$

$$\dot{\mathbf{V}}^n = (11\mathbf{V}^n - 18\mathbf{V}^{n-1} + 9\mathbf{V}^{n-2} - 2\mathbf{V}^{n-3}) / (6\Delta t) \tag{18}$$

The substitution of relations (17)-(18) into equations (14)-(16) yields

$$\overline{\mathbf{H}}\mathbf{U}^n - \mathbf{GT}^n = \overline{\mathbf{L}}^n + \mathbf{W}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) \tag{19}$$

$$\mathbf{O'}^n = \mathbf{G'T}^n - \overline{\mathbf{H}}'\mathbf{U}^n + \overline{\mathbf{L}}'^n + \mathbf{W'}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) + \mathbf{mP}^n \tag{20}$$

$$\overline{\mathbf{H}}''\mathbf{P}^n - \mathbf{G''Q}^n = \overline{\mathbf{L}}''^n + \mathbf{W''O}_e^n (11/6\Delta t) \tag{21}$$

where matrices $\overline{\mathbf{H}}$, $\overline{\mathbf{H}}'$, $\overline{\mathbf{H}}''$ and vectors $\overline{\mathbf{L}}^n$, $\overline{\mathbf{L}}'^n$, $\overline{\mathbf{L}}''^n$ are defined as

$$\overline{\mathbf{H}} = \mathbf{C} + \mathbf{H} + \mathbf{M}(2/\Delta t^2) \tag{22}$$

$$\overline{\mathbf{H}}' = \mathbf{H}' + \mathbf{M}'(2/\Delta t^2) \tag{23}$$

$$\overline{\mathbf{H}}'' = \mathbf{C}'' + \mathbf{H}'' + \mathbf{M}''(11/6\Delta t) \tag{24}$$

$$\overline{\mathbf{L}}^n = \mathbf{M}(5\mathbf{U}^{n-1} - 4\mathbf{U}^{n-2} + \mathbf{U}^{n-3})/\Delta t^2 + \mathbf{S}^n \tag{25}$$

$$\overline{\mathbf{L}}'^n = \mathbf{M}'(5\mathbf{U}^{n-1} - 4\mathbf{U}^{n-2} + \mathbf{U}^{n-3})/\Delta t^2 + \mathbf{S}'^n \tag{26}$$

$$\overline{\mathbf{L}}''^n = (\mathbf{M}''(18\mathbf{P}^{n-1} - 9\mathbf{P}^{n-2} + 2\mathbf{P}^{n-3}) - \mathbf{W}''(18\mathbf{O}'^{n-1} - 9\mathbf{O}'^{n-2} + 2\mathbf{O}'^{n-3}))/(6\Delta t) + \mathbf{S}''^n \tag{27}$$

Equations (19)-(21) can be reordered taking into account the boundary conditions of the problem. Assuming that X stands for the unknown values and Y for prescribed values along the boundary

$$\overline{\mathbf{A}}\,\mathbf{X}^n = \overline{\mathbf{B}}\,\mathbf{Y}^n + \overline{\mathbf{L}}^n + \mathbf{W}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) \tag{28}$$

$$\mathbf{O}'^n = \overline{\mathbf{B}}'\mathbf{Y}^n - \overline{\mathbf{A}}'\mathbf{X}^n + \overline{\mathbf{L}}'^n + \mathbf{W}'\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) + \mathbf{mP}^n \tag{29}$$

$$\overline{\mathbf{A}}''\mathbf{X}''^n = \overline{\mathbf{B}}''\mathbf{Y}''^n + \overline{\mathbf{L}}''^n + \mathbf{W}''\mathbf{O}_e^n(11/6\Delta t) \tag{30}$$

Re-writing equations (28)-(30) in a concise manner, one obtains

$$\mathbf{X}^n = \overline{\mathbf{Y}}^n + \overline{\mathbf{W}}\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) \tag{31}$$

$$\mathbf{O}'^n = \overline{\mathbf{Y}}'^n + \overline{\mathbf{W}}'\left(\mathbf{O}_p^n + \mathbf{mP}^n\right) + \mathbf{mP}^n \tag{32}$$

$$\mathbf{X}''^n = \overline{\mathbf{Y}}''^n + \overline{\mathbf{W}}''\mathbf{O}_e^n \tag{33}$$

where the vectors $\overline{\mathbf{Y}}^n$, $\overline{\mathbf{Y}}'^n$, $\overline{\mathbf{Y}}''^n$ are given by

$$\overline{\mathbf{Y}}^n = \overline{\mathbf{A}}^{-1}\left(\overline{\mathbf{B}}\,\mathbf{Y}^n + \overline{\mathbf{L}}^n\right) \tag{34}$$

$$\overline{\mathbf{Y}}'^n = \overline{\mathbf{B}}'\mathbf{Y}^n - \overline{\mathbf{A}}'\overline{\mathbf{Y}}^n + \overline{\mathbf{L}}'^n \tag{35}$$

$$\overline{\mathbf{Y}}''^n = \overline{\mathbf{A}}''^{-1}\left(\overline{\mathbf{B}}''\mathbf{Y}''^n + \overline{\mathbf{L}}''^n\right) \tag{36}$$

and matrices $\overline{\mathbf{W}}$, $\overline{\mathbf{W}}'$, $\overline{\mathbf{W}}''$ are of the following form

$$\overline{\mathbf{W}} = \overline{\mathbf{A}}^{-1}\mathbf{W} \tag{37}$$

$$\overline{\mathbf{W}}' = \mathbf{W} - \overline{\mathbf{A}}'\overline{\mathbf{W}} \tag{38}$$

$$\overline{\mathbf{W}}'' = \overline{\mathbf{A}}''^{-1}\mathbf{W}''(11/6\Delta t) \tag{39}$$

Based on equation (33) and taking into account the boundary conditions, the pore-pressure vector can be written as indicated by equation (40). Equation (32) can also be re-written as indicated by the following equation, in which $\overline{\mathbf{W'}}_0 = (\overline{\mathbf{W'}} + \mathbf{I})\mathbf{m}$.

$$\mathbf{P}^n = \overline{\mathbf{Y''}}_0^{\,n} + \overline{\mathbf{W''}}_0 \mathbf{O}_e^n \tag{40}$$

$$\mathbf{O'}^n = \overline{\mathbf{Y'}}^n + \overline{\mathbf{W'}}\mathbf{O}_p^n + \overline{\mathbf{W'}}_0 \mathbf{P}^n \tag{41}$$

In order to solve the nonlinear problem, an iterative scheme will be considered to compute the stresses of the model. Once convergence in the stress iterative process is achieved, the displacements, pore-pressures etc. can be computed. The following incremental relation between elastic and elastoplastic stresses is here adopted

$$\Delta\mathbf{O}_p = \mathbf{D}_P \Delta\mathbf{O}_e = \Delta\mathbf{O}_e - \Delta\mathbf{O'} \tag{42}$$

where $\mathbf{D}_P$ is the elastoplastic constitutive matrix and $\Delta\mathbf{O'}$ is the incremental effective stress.

The substitution of equation (40) into equation (41), taking into account relations (42), gives the final expression for the stress problem, as indicated bellow

$$\mathbf{O'}^n = \overline{\overline{\mathbf{Y'}}}^n + \overline{\overline{\mathbf{W'}}}\mathbf{O}_p^n \tag{43}$$

where the vector $\overline{\overline{\mathbf{Y'}}}^n$ and the matrix $\overline{\overline{\mathbf{W'}}}$ are defined as

$$\overline{\overline{\mathbf{Y'}}}^n = (\mathbf{I} - \overline{\mathbf{W'}}_0 \overline{\mathbf{W''}}_0)^{-1} (\overline{\mathbf{Y'}}^n + \overline{\mathbf{W'}}_0 \overline{\mathbf{Y''}}_0^{\,n}) \tag{44}$$

$$\overline{\overline{\mathbf{W'}}}^n = (\mathbf{I} - \overline{\mathbf{W'}}_0 \overline{\mathbf{W''}}_0)^{-1} (\overline{\mathbf{W'}} + \overline{\mathbf{W'}}_0 \overline{\mathbf{W''}}_0) \tag{45}$$

Finally, based on equations (43) and (42), the following iterative algorithm can be developed to solve the stress problem

$$\mathbf{W}_p \, {}^{(k+1)}\Delta\mathbf{O}_e = {}^{(k)}\mathbf{\Psi} \tag{46}$$

$${}^{(k)}\mathbf{\Psi} = \overline{\overline{\mathbf{Y'}}}^n + \mathbf{W}_I \, {}^{(k)}\mathbf{O}_p^n - {}^{(k)}\mathbf{O}_e^n \tag{47}$$

where $\mathbf{W}_p = \mathbf{I} - \mathbf{W}_I \mathbf{D}_P$ and $\mathbf{W}_I = \mathbf{I} + \overline{\overline{\mathbf{W'}}}$. Once the stresses of the model are computed (algorithm (46)-(47)), the other unknowns of the problem can be evaluated by means of equations (33) and (31).

4 NUMERICAL APPLICATIONS

In the present section, linear and nonlinear plane strain problems are considered. The results obtained with the proposed BE formulation are compared with analytical solutions, whenever possible, and other authors/methodologies results, illustrating the viability of the proposed formulation.

4.1 Example 1

On this first example, a one dimensional soil column is analyzed. A sketch of the model is shown in Figure 1(a). The top surface of the column is considered drained and loaded by a time Heaviside type function. The other surfaces of the model are undrained and have null prescribed displacements as indicated by Figure 1(a). The boundary elements and integration cells adopted are depicted in Figure 1(b) (H = 10m).

Two kinds of soil and load amplitudes are considered. The properties of the two different models under analysis are specified below:

(i) Model 1 (de Boer et al., 1993; Soares Jr., 2008, 2010): for the present model, the load amplitude is 3 kN/m2. The physical properties of the soil are: ν = 0.3 (Poisson); E = 14515880 N/m2 (Young Modulus); ρs = 2000 kg/m3 (mass density – solid phase); ρf = 1000 kg/m3 (mass density – fluid phase); μ = 0.33 (porosity); k = 10-6 m4/Ns (permeability). The soil is incompressible.

(ii) Model 2 (Schanz and Cheng, 2000, Soares Jr., 2008, 2010): for the present model, the load amplitude is 1 N/m2. The physical properties of the soil are: ν = 0.2981; E = 254423077 N/m2; ρs = 2700 kg/m3; ρf = 1000 kg/m3; μ = 0.48; k = 3.55x10-9 m4/Ns. The soil is compressible and Ks = 1.1x1010 N/m2 (compression modulus – solid phase); Kf = 3.3x109 N/m2 (compression modulus – fluid phase).

In the case of Model 1, the element length is ℓ = 0.5m and the time-step is given by Δt = 10-3s (β = (cd Δt) / $\ell \approx$ 0.2, where cd is the dilatational wave velocity). For the Model 2 solution, the discretization adopted is analogous to the one depicted in Figure 1(b). However, two different BEM mesh sizes have been considered: ℓ = 0.625m; Δt = 5x10-4s ($\beta \approx$ 1/3) and ℓ = 0.125m; Δt = 1x10-4s ($\beta \approx$ 1/3).

In Figure 2, vertical displacements at point A are depicted, taking into account Model 1. As one can observe, the results are in good agreement with the analytical ones provided by de Boer et al. (1993), as well as with other authors' results (e.g., Diebels and Ehlers, 1996). In Figures 3 and 4, vertical displacements at point A and pore-pressures at point B are depicted, respectively, taking into account Model 2 and different BEM discretization procedures. Once again the results are in good agreement with the ones provided by the semi-analytical processes presented by Dubner and Abate (1968), and Schanz and Cheng (2000).

4.2 Example 2

In this second example a 2-D soil strip is analyzed. A sketch of the model is depicted in Figure 5(a) (a = 5m, b = 10m and c = 1m). The FEM and BEM meshes adopted for the problem solution are shown in Figure 5(b) (symmetry is taken into account). For the FEM analysis, 100 linear quadrilateral finite elements were adopted; 200 linear triangular integration cells and 40 linear boundary elements were adopted for the BEM analysis.

The soil strip is loaded as indicted in Figure 5(a). The surface under the applied load is considered undrained, as well as the vertical and bottom surfaces. The properties of the soil are: ν = 0.2; E = 107

N/m2; ρs = 2538.5 kg/m3; ρf = 1000 kg/m3; μ = 0.35; k = 10-7 m4/Ns. The soil fluid phase is compressible and Kf = 3.3x109 N/m2. A perfectly plastic material obeying the Mohr-Coulomb yield criterion is assumed, where c = 200 N/m2 (cohesion) and ϕ = 10o (internal friction angle).

The time-step considered for both FEM and BEM solutions is Δt = 2x10-3s. Vertical displacement results at point A (see Figure 5(a)) are depicted in Figure 6, taking into account linear and nonlinear analyses. In Figure 7, the pore-pressure distribution over the FEM and BEM meshes are depicted at time t = 0.8s. Good agreement between the FEM and BEM solutions is seen to be obtained.

5 CONCLUSIONS

A time domain BEM formulation for the dynamic analysis of porous media was presented. Time independent fundamental solutions were employed originating an approach that requires domain discretization, but, on the other hand, is quite amenable to deal with elastoplastic models. The general pore-dynamic equations were initially discussed, however, the numerical methodology developed was based, as it is usual, on the simplified equations that are obtained when contributions of high frequency components of the spectrum can be disregarded. A detailed description of the BEM (integral and matrix) equations and solution procedures, which resulted from a direct coupling approach, was presented. Accurate results were obtained for the examples considered, showing that the coupling methodology discussed here leads to robust algorithms. The present paper extends the applicability of boundary elements to model complex pore-dynamic problems.

6 REFERENCES

BENALLAL, A.; BOTTA, A.; VENTURINI, W. S. Consolidation of elastic-plastic saturated porous media by the boundary element method. **Computer Methods in Applied Mechanics and Engineering,** n. 197, p. 4626-4644, 2008.

BIOT, M. A. General theory of three-dimensional consolidation. **Journal of Applied Physics,** n.12, p. 155-164, 1941.

BIOT, M. A. The theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low frequency range. **J. Acoust. Soc. Am.,** n. 28, p. 167-178, 1956a.

BIOT, M. A. The theory of propagation of elastic waves in a fluid-saturated porous solid, II. Higher frequency range. **J. Acoust. Soc. Am.**, n.28, p. 179-191, 1956b.

BIOT, M. A. Mechanics of deformation and acoustic propagation in porous media. **Journal of Applied Physics.**, n. 33, p. 1482-1498, 1962.

DE BOER, R. **Theory of porous media – highlights in the historical development and current state.** Berlin: Springer, 1998.

DE BOER, R.; EHLERS, W.; LIU, Z. One-dimensional transient wave propagation in fluid saturated incompressible porous media**. Arch. Appl. Mech.,** n. 63, p. 59-72, 1993.

BONNET, G. Basic singular solution for a poroelastic medium in the dynamic range**. J. Acoust. Soc. Am.,** n. 82, p. 1758-1762, 1987.

BOUTIN, C.; BONNET, G.; BARD, P. Y. Green functions and associated sources in infinite and stratified poroelastic media. **Geophys. J. Roy. Astr. Soc**., n. 90, p. 521-550, 1987.

CAVALCANTI, M. C., TELLES, J. C. F. Biot's consolidation theory – application of BEM with time independent fundamental solutions for poro-elastic saturated media. **Engineering Analysis with Boundary Elements**, n. 27, p. 145-157, 2003.

CHEN, J. Time domain fundamental solution to Biot's complete equations of dynamic poroelasticity. Part I: two-dimensional solution**. International Journal of Solids and Structures**, n. 31, p. 1447-1490, 1994a.

CHEN, J. Time domain fundamental solution to Biot's complete equations of dynamic poroelasticity. Part II: three-dimensional solution". **International Journal of Solids and Structures**, n. 31, p. 162-202, 1994b.

CHEN, J.; DARGUSH, G. F. Boundary element method for dynamic poroelastic and thermoelastic analysis. **International Journal of Solids and Structures**, n. 32, p. 2257-2278, 1995.

CHENG, A. H. D., BADMUS, T., BESKOS, D. E. Integral equations for dynamic poroelasticity frequency domain with BEM solution**. J. Engng. Mec. ASCE**, n. 117, p. 1136-1157, 1991.

CODA, H. B.; VENTURINI, W.S. 3-Dimensional transient BEM analysis. **Computer and Structures**, n. 56, p. 751-768, 1995.

DIEBELS, S.; EHLERS, W. Dynamic analysis of fully saturated porous medium accounting for geometrical and material non-linearities. **International Journal for Numerical Methods in Engineering**, n. 49, p. 833-848, 1996.

DOMINGUEZ, J. Boundary element approach for dynamic poroelastic problems. **International Journal for Numerical Methods in Engineering.,** n.35, p. 307-324, 1992.

DUBNER H.; ABATE J. Numerical inversion of Laplace transforms by relating them to the finite Fourier cosine transform. **Journal of the Association for Computing Machinery**, n. 15, p. 115-123, 1968.

HOUBOLT, J. C. A recurrence matrix solution for the dynamic response of elastic aircraft. **Journal of the Aeronautical Sciences**, n. 17, p. 540-550, 1950.
LI, X.; HAN, X.; PASTOR, M. An iterative stabilized fractional step algorithm for finite element analysis in saturated soil dynamics. **Computational Methods in Applied Mechanics and Engineering**, n. 192, p. 3845-3859, 2003.

LIGGETT, J. A.; LIU, P. L. F. **The BIEM for Porous Media Flow**. London: Allen & Unwin, 1983.

LUBICH, C. Convolution quadrature and discretized operational calculus I. **Numerische Mathematik**. n. 52, p. 129-145, 1988a.

LUBICH, C. 1988b. Convolution quadrature and discretized operational calculus II. **Numerische Mathematik,** n. 52, p. 413-425.

MANOLIS, G.; BESKOS, D.E. Integral formulation and fundamental solutions of dynamic poro-elasticity and thermolasticity. **Acta Mechanica**, n. 76, p. 89-104, 1989.

MANSUR, W. J. **A Time-Stepping Technique to Solve Wave Propagation Problems Using the Boundary Element Method**, Ph.D. (Thesis) – University of Southampton, England, 1983

NORRIS, A. N. Radiation from a point source and scattering theory in a fluid-saturated porous solid. **J. Acoust. Soc. Am**, n. 77, p. 2012-2023, 1985.

PREDELEANU, M. Development of boundary element method to dynamic problems for porous media. **Appl. Math. Modeling**., n. 8, p. 378-382, 1984.

SCHANZ, M. Application of 3D time domain boundary element formulation to wave propagation in poroelastic solids. **Engineering Analysis with Boundary Elements**. n. 25, p. 363-376, 2001a

SCHANZ, M. **Wave propagation in viscoelastic and poroelastic continua: a boundary element approach**. Berlin, New York: Springer, 2001b.

SCHANZ, M.; CHENG, A. H. D. Transient wave propagation in a one-dimensional poroelastic column. **ACTA Mechanica**, n. 145, p. 1-18, 2000.

SOARES JR, D. A time-domain FEM approach based on implicit Green's functions for the dynamic analysis of porous media. **Computational Methods in Applied Mechanics and Engineering**, n. 197, p. 4645-4652, 2008.

SOARES JR, D. Dynamic Analysis of porous media considering unequal phase discretization by meshless local Petrov-Galerkin formulations. **CMES – Computer Modeling in Engineering & Sciences**, n. 61, p. 177-200, 2010.

SOARES JR, D.; CARRER, J. A. M.; MANSUR, W. J. Nonlinear elastodynamic analysis by the BEM: an approach based on the iterative coupling of the D-BEM and TD-BEM formulations. **Engineering Analysis with Boundary Elements.** 29, 761-774. 2005.

SOARES JR, D.; TELLES, T. C. F.; MANSUR, W. J. A time-domain boundary element formulation for the dynamic analysis of non-linear porous media. **Engineering Analysis with Boundary Elements**, n. 30, p. 363-370, 2006.

TELLES, J. C. F. **The boundary element method applied to inelastic problems.** Berlin: Springer-Verlag, 1983.

VENTURINI, W. S. **Boundary element method in geomechanics**. Berlin: Springer-Verlag, 1984.

VENTURINI, W. S.; BOTTA, A. S.; BENALLAL, A. Domain boundary element formulation for poro-elastic solids. In: INTERNATIONAL CONFERENCE ON BOUNDARY ELEMENT TECHNIQUES, 6., Montreal. **Advances in boundary element techniques,** 6., 2005. v.1.

VENTURINI, W. S.; BOTTA, A. S.; WUTZOW, W. W. Poroelastic analysis using boundary element method considering fluid compressibility. In: IBERIAN LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING, Vitória. Editora da Universidade Federal do Espirito Santo, 2005. v.1.

VENTURINI, W. S.; WUTZOW, W. W.; BOTTA, A. S.; BENALLAL, A. Análise de sólidos poro-elásticos enrijecidos utilizando o método dos elementos de contorno. In: IBERIAN LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING, Vitória. Editora da Universidade Federal do Espirito Santo, 2005. v.1.

WIEBE, T.; ANTES, H. A time domain integral formulation of dynamic poroelasticity. **Acta Mec.** n. 90, p. 125-137, 1991.

ZIENKIEWICZ, O. C.; CHAN, A. H. C.; PASTOR, M.; PAUL, D. K.; SHIOMI, T. Static and dynamic behavior of soils: a rational approach to quantitative solutions, I. Fully saturated problems. **Proc. R. Soc. Lond. A**, n. 429, p. 285-309, 1990.

ZIENKIEWICZ, O. C.; SHIOMI, T. Dynamic behavior of saturated porous media: the generalized Biot formulation and its numerical solution. **International Journal for Numerical Methods in Engineering**, n. 8, p. 71-9, 1984.

APPENDIX A. FUNDAMENTAL SOLUTIONS

The fundamental solution components that appear in equations (7)-(9), taking into account plane strain problems, are defined as

$$p^*(X;\xi) = \ln(r)/(2\pi) \tag{A1}$$

$$q^*(X;\xi) = r_n/(2\pi r) \tag{A2}$$

$$u_{ik}^*(X;\xi) = \frac{-1}{8\pi(1-\nu)G}\left[(3-4\nu)\ln(r)\delta_{ik} - r,_i\, r,_k\right] \tag{A3}$$

$$\tau_{ik}^*(X;\xi) = \frac{-1/r}{4\pi(1-\nu)}\left\{\left[(1-2\nu)\delta_{ik} + 2r,_i\, r,_k\right] r_n - (1-2\nu)(r,_i\, n_k - r,_k\, n_i)\right\} \tag{A4}$$

$$\varepsilon_{ikj}^*(X;\xi) = \frac{-1/r}{8\pi(1-\nu)G}\left[(1-2\nu)(r,_j\,\delta_{ik} - r,_k\,\delta_{ij}) - r,_i\,\delta_{kj} + 2r,_i\, r,_k\, r,_j\right] \tag{A5}$$

$$u_{ikj}^*(X;\xi) = \frac{1/r}{4\pi(1-2\nu)}\left[(1-2\nu)(r,_k\,\delta_{ij} + r,_i\,\delta_{kj} - r,_j\,\delta_{ik}) + 2r,_i\, r,_k\, r,_j\right] \tag{A6}$$

$$\begin{aligned}\tau_{ikj}^*(X;\xi) = {}& \frac{G/r^2}{2\pi(1-\nu)}\left\{2r_n\left[(1-2\nu)r,_j\,\delta_{ik} + \nu(r,_k\,\delta_{ij} + r,_i\,\delta_{kj}) + 4r,_i\, r,_k\, r,_j\right] + \right.\\ & \left. + 2\nu(r,_k\, r,_j\, n_i + r,_i\, r,_j\, n_k) - (1-4\nu)n_j\delta_{ik} + (1-2\nu)(2r,_i\, r,_k\, n_j + n_k\delta_{ij} + n_i\delta_{kj})\right\}\end{aligned} \tag{A7}$$

$$\begin{aligned}\varepsilon_{ikjl}^*(X;\xi) = {}& \frac{G/r^2}{4\pi(1-\nu)}\left\{(1-2\nu)(\delta_{ij}\delta_{ik} + \delta_{kj}\delta_{li} - \delta_{ik}\delta_{jl} + 2r,_l\, r,_j\,\delta_{lk}) + \right.\\ & \left. + 2\nu(r,_k\, r,_j\,\delta_{li} + r,_l\, r,_i\,\delta_{kj} + r,_l\, r,_k\,\delta_{ij} + r,_l\, r,_j\,\delta_{kl}) + 2r,_i\, r,_k\,\delta_{jl} - 8r,_i\, r,_j\, r,_k\, r,_l\right\}\end{aligned} \tag{A8}$$

where $r_n = \partial r/\partial n$ and n is the outward unit vector normal to the Γ boundary. The free term $g_{ik}(X_{jl})$ is given by

$$g_{ik} = \frac{-1}{8(1-\nu)}\left[2X_{ik} + (1-4\nu)X_{jj}\delta_{ik}\right] \tag{A9}$$

CAPTION TO THE FIGURES

Figure 1 – 1-D problem: (a) sketch of the model; (b) boundary element and internal cell discretization.

Figure 2 – Displacement at point A for the incompressible soil column.

Figure 3 – Displacement at point A for the compressible soil column: (a) $\Delta t = 5x10^{-4}s$ and $\ell = 0.625m$; (b) $\Delta t = 1x10^{-4}s$ and $\ell = 0.125m$.

Figure 4 – Pore-pressure at point B for the compressible soil column: (a) $\Delta t = 5x10^{-4}s$ and $\ell = 0.625m$; (b) $\Delta t = 1x10^{-4}s$ and $\ell = 0.125m$.

Figure 5 – 2-D problem: (a) sketch of the model; (b) FEM and BEM discretizations.

Figure 6 – Displacement at point A considering linear and nonlinear analyses.

Figure 7 – Pore-pressure distribution for FEM and BEM at *t* = 0.8*s*: (a) elastic analysis; (b) elastoplastic analysis.

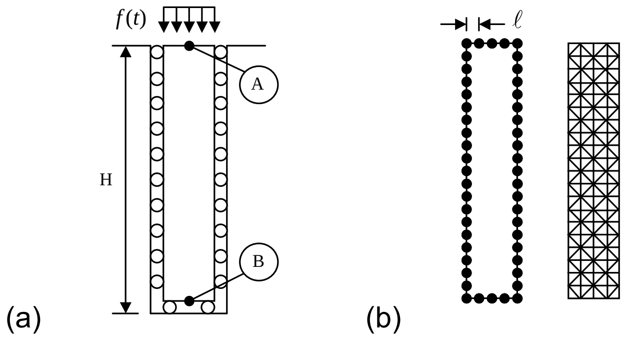


Figure 1.

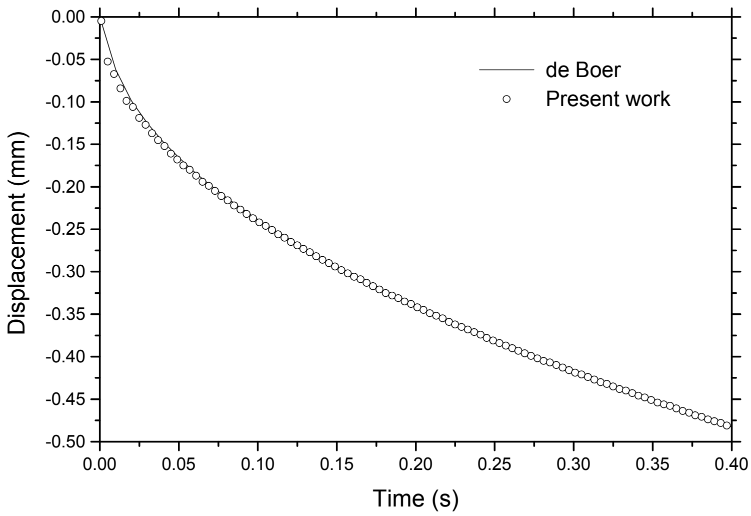
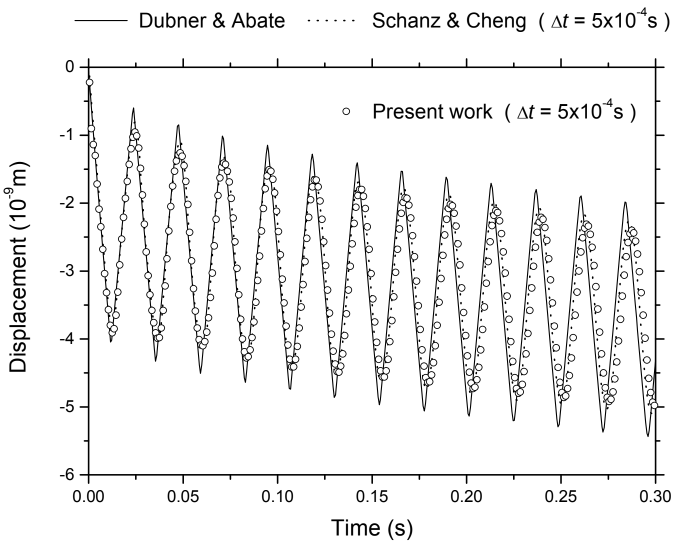


Figure 2.

(a)


Dubner & Abate
Schanz & Cheng (Δt = 5x10^-4 s)
Present work (Δt = 5x10^-4 s)
Displacement (10^-9 m)
0
-1
-2
-3
-4
-5
-6
0.00
0.05
0.10
0.15
0.20
0.25
0.30
Time (s)

(b)

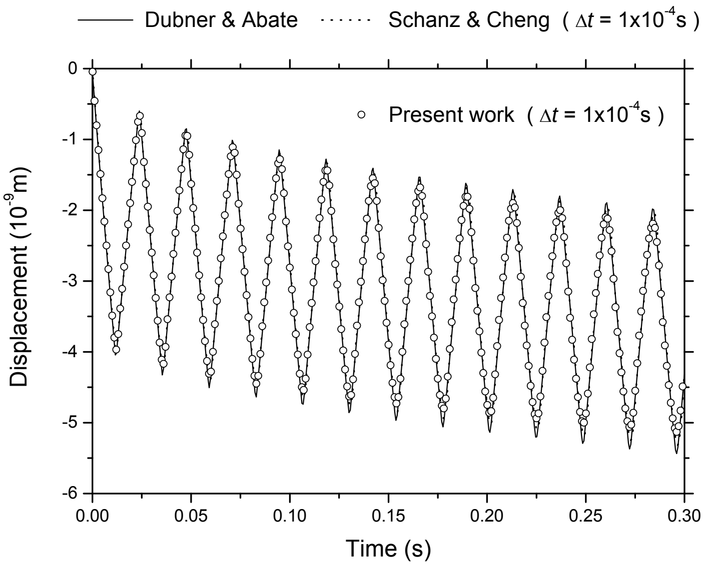

Dubner & Abate
Schanz & Cheng (Δt = 1x10^-4 s)
Present work (Δt = 1x10^-4 s)
Displacement (10^-9 m)
0
-1
-2
-3
-4
-5
-6
0.00
0.05
0.10
0.15
0.20
0.25
0.30
Time (s)

Figure 3.

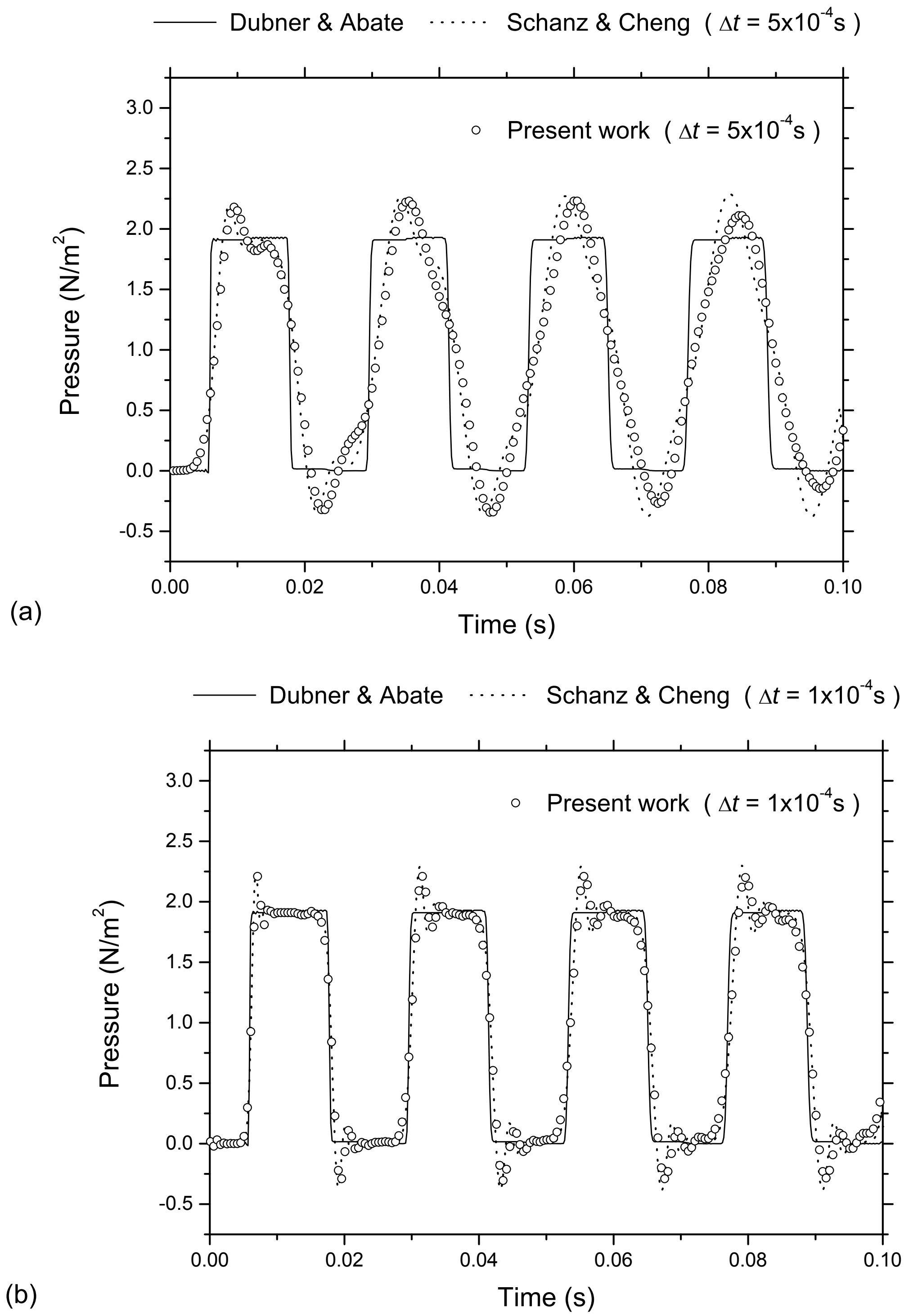

Dubner & Abate
Schanz & Cheng (Δt = 5x10^-4 s)
Present work (Δt = 5x10^-4 s)
Pressure (N/m2)
Time (s)
(a)
Dubner & Abate
Schanz & Cheng (Δt = 1x10^-4 s)
Present work (Δt = 1x10^-4 s)
Pressure (N/m2)
Time (s)
(b)

Figure 4.

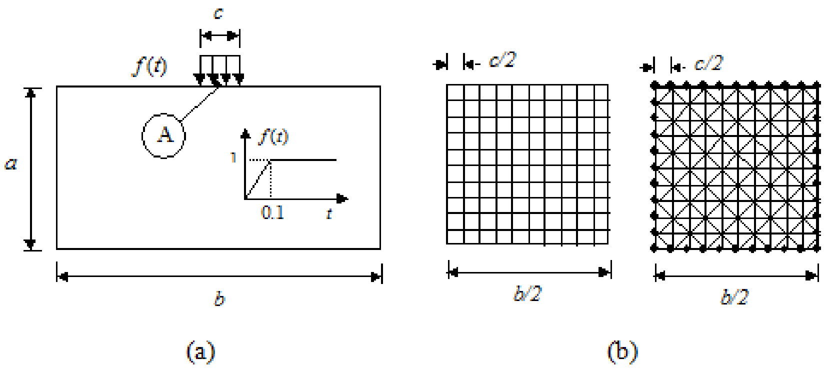


Figure 5.

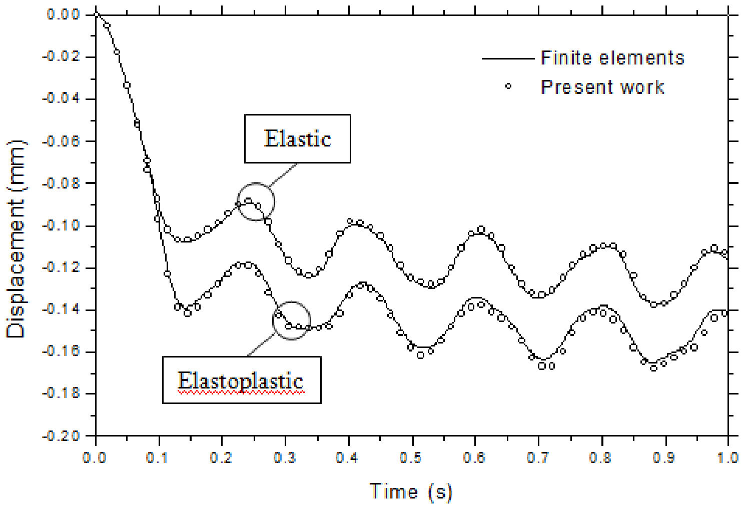


Figure 6.

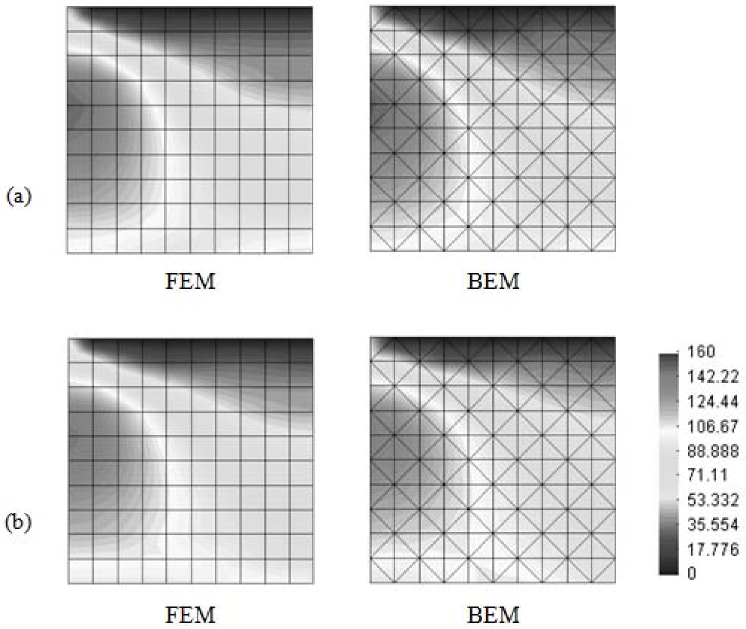


Figure 7.